Instability of Plane Couette Flow

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Abstract

The energy gradient theory has been proposed with the aim of better understanding the mechanism of flow transition from laminar flow to turbulent flow. In this theory, it is suggested that the transition to turbulence depends on the relative magnitudes of the energy gradient amplifying the disturbance and the viscous friction damping that disturbance. For a given flow geometry and fluid properties, when the maximum of *K* (the ratio of the energy gradient in the transverse direction to that in the streamwise direction) in the flow field is larger than a certain critical value, it is expected that instability would occur for some initial disturbances. In this paper, using the energy analysis, the equation for calculating *K* for plane Couette flow is derived. It is demonstrated that the critical value of *K* at subcritical transition is about 370 for plane Couette flow. This value is about the same as for plane Poiseuille flow and pipe Poiseuille flow (385-389). Therefore, it is concluded that the critical value of *K* at subcritical transition is about 370-389 for wall bounded parallel shear flows which include both pressure and shear driven flows.

Keywords: Flow instability; Plane Couette flow; Energy gradient; Energy loss; Critical condition

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47.20.Gv	Viscous instability

Introduction

Although more than a century has passed since the pioneering work of Reynolds (1883) was done, flow transition from laminar flow to turbulence is still not completely understood [1-5]. In practice, the understanding of turbulence transition and generation has great significance for basic sciences and many engineering fields. This issue is intricately related to the instability problem of the base flow subjected to some imposed disturbances [1-2].

In the past, several stability theories have been developed to describe the mechanism of flow instability. These are: (1) The linear stability theory, which can be back to Rayleigh (1880) is a widely used method and has been applied to some problems [6]. For Taylor-Couette flow and Rayleigh-Bernard convective problem, it agrees well with experimental data. However, this theory fails when used for wall bounded parallel flows such as plane Couette flow, plane Poiseuille flow and pipe Poiseuille flow. (2) The energy method (Orr, 1907) is another mature method for estimating flow instability [7]. However, full agreement could not be obtained between the theoretical predictions and the experiment data. (3) The weak nonlinear stability theory (Stuart, 1960) emerged in the last century (see [8]). (4) The secondary instability theory (Herbert et al, 1988) was developed most recently, and it explains some of flow transition phenomena better than the other earlier theories (see [9]). However, there are still significant discrepancies between the predictions obtained using this method and experimental data; particularly at transition.

Dou [10] proposed an energy gradient theory with the aim of clarifying the mechanism of flow transition from laminar to turbulence. For plane Poiseuille flow and Hagen-Poiseuille flow, this theory yields results consistent with experimental data. This method is also used to explain the mechanism of instability of inflectional velocity profile for viscous flow and is valid for pressure driven flows. However, for the shear driven flows such as plane Couette flow, it has yet to be tested.

In this paper, the stability of plane Couette flow is further studied (Fig.1) using the energy gradient theory. It will be shown that the energy gradient theory is also suitable for shear driven flows. The critical value of the energy gradient parameter is about the same as that for Poiseuille flows.

Energy Gradient Theory

In the energy gradient theory [10], it is suggested that the instability of the flow depends on the relative magnitude of the energy gradients in transverse direction and streamwise direction. It is suggested that the energy gradient in the transverse direction has a potential to amplify a velocity disturbance, while the viscous friction loss in the streamwise direction can resist and absorb this disturbance energy. The transition to turbulence therefore depends on the relative magnitude of the two roles from energy gradient amplifying and viscous friction damping to the initial disturbance. Based on such, a dimensionless parameter, *K* (the ratio of the energy gradient in the transverse direction to that in the streamwise direction), can be written as,

$$K = \frac{\partial E / \partial n}{\partial E / \partial s} . \tag{1}$$

Here, $E = p + \frac{1}{2}\rho V^2 + \rho g \xi$ is the total energy in unit volume fluid for incompressible flows with ξ as the coordinate perpendicular to the ground, *n* denotes the direction normal to the streamwise direction and *s* denotes the streamwise direction. The occurrence of instability depends upon the magnitude of this dimensionless parameter *K,* and the critical condition is determined by the maximum value of *K* in the flow. For a given flow geometry and fluid properties, when the maximum of *K* in the flow field is larger than a critical value *Kc*, it is expected that instability would occur for certain initial disturbance. The analysis for Poiseuille flows showed that the transition to turbulence is due to the energy gradient and the disturbance amplification [10], rather than a linear eigenvalue instability type [11-12].

For Poiseuille flows, Dou [10] demonstrated that the energy gradient theory has led to a consistent value of Kc at the subcritical condition of transition determined by experiments. It is found that Kc=385~389 at the subcritical condition determined by experiments for both plane Poiseuille flow and pipe Poiseuille flow, as pointed in Table 1. The most unstable position for plane Poiseuille flow and pipe Poiseuille flow occurs at y/h= ± 0.5774 and r/R=0.5774, respectively. These said locations have been confirmed by experiments and simulations [10].

In plane Couette flow, the streamwise energy gradient (energy loss) for unit volume fluid could not be obtained directly from the Navier-Stokes equation as for Poiseuille flows. Using the energy analysis, the equation for calculating the parameter K is derived below as for the plane Couette flow.

Energy Gradient Theory for Plane Couette Flow

In plane Couette flow, the viscous term $\mu \nabla^2 \mathbf{u}$ in Navier-Stokes equation is zero, and the fluid energy $p + \frac{1}{2}\rho V^2$ in unit volume is constant along the streamwise direction. This is not to say that there is no energy loss due to friction in the flow. The friction loss must still occur since this is a viscous flow (A zero energy loss only occurs in inviscid flow). The energy magnitude is kept constant because the energy loss due to viscous friction is exactly compensated by the energy input to the flow by the moving walls. The work done on the flow by the wall is balanced by the energy loss in the flow.

The velocity distribution for plane Couette flow can be obtained by solving the Navier-Stokes equation and applying the boundary conditions, as in [3].

$$u = ky = \frac{U}{h} y \, . \tag{2}$$

Here, $k = \partial u / \partial y$ is the shear rate and is determined by $k = U/h$. The velocity profile is shown in Fig.1. The shear stress is calculated as

$$\tau = \mu \partial u / \partial y = \mu k \, . \tag{3}$$

The energy gradient in the transverse direction is calculated by,

$$\rho V \frac{\partial V}{\partial y} = \rho k y \cdot k = \rho k^2 y \, . \tag{4}$$

Taking an element in the fluid layer as shown in Fig.2, the work done to the element by the upper layer is

$$A_1 = F \Delta x = \tau \cdot \Delta x \cdot \Delta z \cdot (u + \Delta u) dt \, .$$

The work done on the lower layer by the element is

$$A_2 = \tau \cdot \Delta x \cdot \Delta z \cdot u dt \, .$$

Therefore, the net work done on the element is given as

$$\Delta A = A_1 - A_2 = \tau \cdot \Delta x \cdot \Delta z \cdot \Delta u dt \, .$$

The fluid volume passing through *dy* depth in *dt* time is

$$\Delta Q = \Delta y \cdot \Delta z \cdot u dt \, .$$

Hence, the energy consumed by the element in unit time and unit volume (Fig.2) is

$$\Delta E = \frac{\Delta A}{\Delta Q} = \frac{\tau \Delta x \Delta z \Delta V dt}{\Delta y \Delta z V dt} = \frac{\tau \Delta x \Delta u}{\Delta y \cdot u} \, . \tag{5}$$

Then, the gradient of the consumed energy (energy loss gradient) in streamwise direction is,

$$\frac{\Delta E}{\Delta x} = \frac{\tau}{u} \frac{\Delta u}{\Delta y} \, . \tag{6}$$

Thus, we have

$$\frac{\partial E}{\partial x} \equiv \frac{\tau}{u} \frac{du}{dy} \, . \tag{7}$$

Introducing Eq.(4) and (7) into Eq.(1), then the ratio of the energy gradient in the two directions, K, can be written as,

$$K = \frac{\rho u(\partial u / \partial y)}{\partial E / \partial x} = \frac{\rho k k y}{\frac{\tau}{u}\frac{du}{dy}} = \frac{\rho U h}{\mu}\frac{y^2}{h^2} = \mathrm{Re}\frac{y^2}{h^2}, \tag{8}$$

where $\mathrm{Re} = \rho U h / \mu$ is the Reynolds number. It can be seen that the magnitude of K is proportional to Re at any location in the flow field. K is a quadratic function of y/h across the channel width (see Fig.3). There is no maximum in the channel as that for Poiseuille flows [10]. It reaches its maximum only on the walls ($y = \pm h$),

$$K_{\max} = \frac{\rho U h}{\mu} = \mathrm{Re}. \tag{9}$$

From the energy gradient theory, the flow is expected to be more unstable in the area of high value of K than that in the area of low value of K. The first instability should be associated with the maximum of K, K_{max}, in the flow field for a given disturbance. That is, the position of maximum of K is the most unstable position. For a given flow disturbance, there is a critical value of K_{max} over which the flow becomes unstable. Now, it is difficult to directly predict this critical value by theory. However, it can be determined using available experimental data.

Flow type	Re expression	Eigenvalue analysis, Re_c	Experiments, Re_c	K_{max} at Re_c (from experiments), $\equiv$Kc
Pipe Poiseuille	$\mathrm{Re} = \rho U D / \mu$	Stable for all Re	2000	385
Plane Poiseuille	$\mathrm{Re} = \rho U L / \mu$	7696	1350	389
	$\mathrm{Re} = \rho u_0 h / \mu$	5772	1012	389
Plane Couette	$\mathrm{Re} = \rho U h / \mu$	Stable for all Re	370	370

Table 1 Comparison of the critical Reynolds number and the critical value of the energy gradient parameter K_{max} for plane Poiseuille flow and pipe Poiseuille flow as well as for plane Couette flow. U is the averaged velocity, u_0 the velocity at the mid-plane of the channel, D the diameter of the pipe, L=2h, h the half-width of the channel for plane Poiseuille flow and plane Couette flow.

Discussions

Instability mechanism and disturbance amplification

The development of a disturbance is subjected to the governing equations and the boundary and initial conditions. Thus, the flow stability depends on the distribution of K in the flow field and the initial disturbance provided to the flow. On the other hand, we should distinguish between the disturbance in laminar state with the velocity fluctuation in turbulent state. The laminar flow is of a completely different flow state from turbulent flow regarding to the disturbance. The place where the disturbance is the largest in laminar flow is not necessarily the same as that where the turbulent stresses is largest in the corresponding turbulent state. At the wall, the capacity of the base flow to amplify a disturbance is largest owing to the largest magnitude of K (Fig.3). However, the flow disturbances at the wall should be vanishing due to no-slip condition. Therefore, it is likely that the most dangerous position is not directly at the wall, but at a location (very) near the wall where the initial disturbance is present and yet K has a large magnitude. Thus, a small disturbance could be easily amplified by the large energy gradient at such position. Therefore, the layer near the wall is the most dangerous position. Fig.4 shows the measurements by Bottin et al (1998) for plane Couette flow during the process leading to the formation of the turbulent spot [13]. Three slices of profiles are sketched within a turbulent spot in plane Couette flow. This picture was taken for the flow near the critical condition (Re=340). The profile on the right side is at the edge of the spot and is in the initial instability stage. It is seen that the flow oscillation first occurs near one of the moving walls. From this figure, it is also observed that the process of flow transition is not symmetrical relative to the channel width which might be subjected to the influence of other factors (for example, gravity force). This experiment indicates that role of the K_{max} is dominating, and further lend credibility to the theory.

Comparison with experiments at critical condition

Lundbladh and Johansson (1991)'s direct numerical simulation produced a critical condition of Rec=375 for plane Couette flow [14]. For this flow, Hegseth et al [15] observed an intermittent turbulent state which occurs in the range of Re of 380--450. Below this range, the laminar state is stable to finite amplitude perturbation and above this range the entire flow domain is turbulent. For 380<Re<450 and after a perturbation, the dynamical regime shows a fluctuating mixture of laminar and turbulent domains which is reminiscent of spatiotemporal intermittency. This result showed that the minimum Re for the transition with finite amplitude disturbance is about 380. Tillmark and Alfredsson [16], Davidud et al [17], and Malerud et al [18] carried out some experiments for turbulent transition for plane Couette flows using flow visualization techniques. All of these experiments showed that the critical condition occurs at about $Rec=370\pm10$. Although the subsequent experimental results showed a lower critical Reynolds number (325~380) [19-20], this does not detract the comparisons carried out here. Using the experimental data Rec=370, we obtain Kc=370 from Eq.(9) at the critical condition determined by experiments below which no turbulence occurs (see Table 1). The most unstable position is at the walls. This critical value of Kc=370 is near to the value for Poiseuille flows, 385~389. The small difference in the value obtained is subjected to the uncertainty of the critical condition in experiments. For example, the determination of transition is deduced from the abrupt change in the drag coefficient as found by Patel and Head [21], while the flow visualization method is used in [16-18]. These results demonstrate that the critical value of Kmax for wall bounded parallel flows including both pressure driven and shear driven flows is about 370~389. This consistency also suggests that the mechanisms of instabilities in wall bounded parallel shear flow are perhaps the same. They are all dominated by the transverse energy gradient and the streamwise flow energy loss. The results obtained in this study provide further basis for better understanding of the mechanism of instability and transition to turbulence in parallel shear flows.

Flux of vorticity rather than vorticity

In turbulence modelling, the turbulent stress is generally modelled in terms of the velocity gradient of mean flows or the strain-rate-tensor. The magnitude of velocity gradient is an indication of the strength of turbulent stress in some cases. Thus, it may be deduced that the turbulent transition might be related to the magnitude of the velocity gradient. However, from the energy gradient theory, the magnitude of the velocity gradient or vorticity is not the dominating factor influencing the transition. On the other hand, the flux of vorticity ($u\partial u/\partial y$ in 2D parallel flow case) or the energy gradient is the governing factor. Fig.5 depicts the development of plane Couette flow with the increase of the channel width. In these three cases, the shear rate (velocity gradient) is kept the same and the width of the channel is allowed to vary. With the increase of the channel width, Re increases. As we can see from experiments, the flow kept to its laminar in

(a) and (b) when the Re is lower than the critical Reynolds number. The flow in (c) becomes turbulent when the Re is higher than a critical value. It is therefore suggested that the transition to turbulence is not due directly to the influence of the velocity gradient. This is attributed to the roles of the transverse energy gradient (or flux of vorticity in 2D parallel flow) and the streamwise energy loss from energy gradient theory.

Discussion and Conclusion

In this paper, the instability of plane Couette flow is studied. The expression of K, the ratio of the energy gradient in transverse direction to that in streamwise direction for plane Couette flow, is derived using energy analysis. At the experimentally determined critical condition, it is demonstrated that the critical value of $K_{\max}$ is about 370 (or 325~370) for plane Couette flow and the most unstable position is at the walls. It is interesting to note that based on the critical conditions determined by experiments, the critical value of K for Couette flow is about the same as that for plane Poiseuille flow and pipe Poiseuille flow (385-389). Therefore, it is suggested that the critical value of $K_{\max}$ at transition is about 370-389 (or 325~389) for wall bounded parallel flows which include both pressure and shear driven flows. These results show that the energy gradient theory is suitable for both pressure and shear driven flows. This theory may provide a basis for the modelling and prediction of the transition process. This is especially relevant since the shear driven flow and pressure driven flow are very different, and yet the mechanism for transition to turbulence bears much similarities. The marked difference between the shear driven and pressure driven flows lies in the energy transmission process: in the former, the energy is transported to the wall from the core, but in the latter it is transported to the core from the wall. This may lead to different turbulent stress distribution near the wall as simulated by Bech et al. [22]. The energy gradient theory is also demonstrated valid for Taylor-Couette flow between concentric rotating cylinders [23] more recently. Excellent agreement is achieved with all the experimental data in literature.

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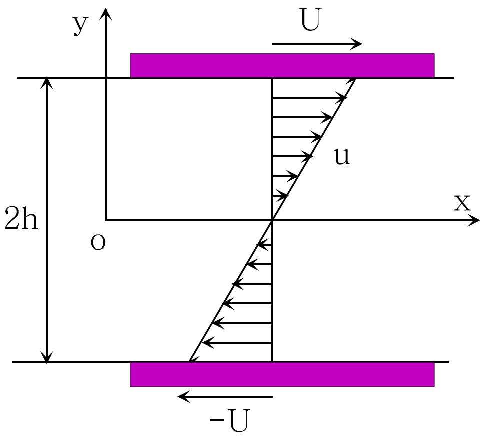

Fig.1 Plane Couette flow.

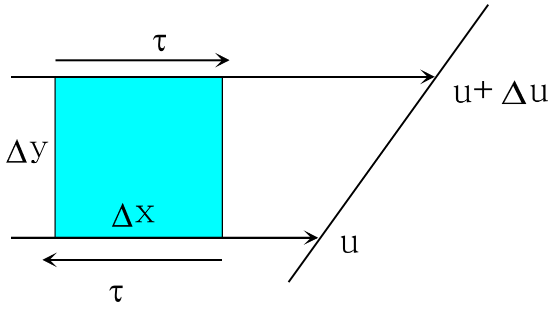

Fig.2 A cubic fluid element. Δz is perpendicular to x-y plane.

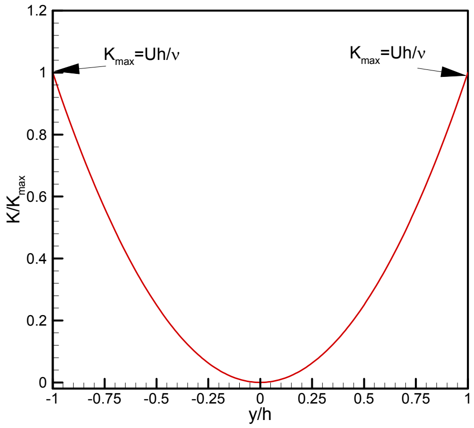

Fig.3 Distribution of *K* versus the channel width.

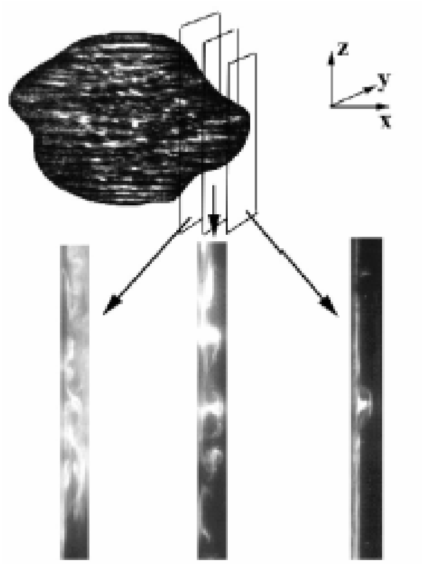

Fig.4 Sectional views of the flow at the border of a turbulent spot in three x=constant planes for Re=340 for plane Couette flow. x is in the streamwise direction, y is in transverse direction, and z is in the spanwise direction (Bottin et al 1998; Courtesy of Dauchot).

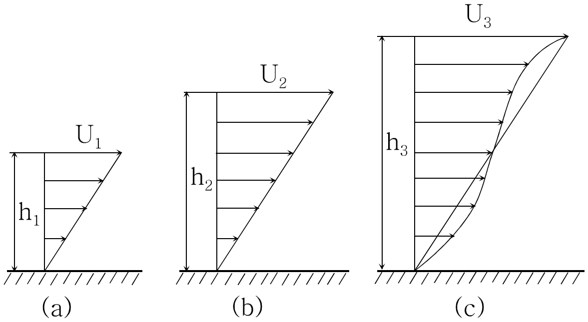

Fig.5 Development of plane Couette flow with increasing channel width. The shear rate is the same for three cases. With increasing of the channel width, Re increases. The flow is laminar for (a) and (b) cases, and it becomes turbulent for (c) with the increasing Re.