Compatible Differential Constraints to an Infinite Chain of Transport Equations for Cumulants *

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Abstract

We study an infinite chain of transport equations for cumulants which appears in modeling the dynamics of a momentumless turbulent planar wake. The method of compatible differential constraints for formulating its integrability properties is applied. The compatibility conditions obtained make it possible to realize a reduction of the original infinite chain of transport equations and to present an algorithm for calculating cumulants of arbitrary order.

Keywords: differential constraints, infinite chain of transport equations for cummulants, invariant manifold, momentumless turbulent plane wake, selfsimilar solution
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1 Introduction

For a complete statistical description of the hydrodynamical characteristic fields in a turbulent flow, it is necessary to obtain the complete multi-dimensional joint distributions of probability for the values of these characteristics at every possible ensemble of space-time points. However the definition of such multi-dimensional distributions is a highly complex problem. Moreover these distributions are often inconvenient for applications because solutions for the complex equations are difficult to obtain. Therefore, in practice calculations are restricted to employing only some simpler statistical parameters of propability distributions such as moments, cumulants and some of their combinations. We remind

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that near zero, the logarithm of the characteristic function (Fourier transformation of the probability density function (PDF)) can be presented as a Taylor series:

$$\ln \chi(\vec{q}) = \sum_{n,m,k=0}^{r} \mathbf{C}_{n+m+k} \frac{(iq_1)^n}{n!} \frac{(iq_2)^m}{m!} \frac{(iq_3)^k}{k!} + O((\vec{q})^r),$$

$$\chi(\vec{q}) = \int_R P(\vec{u}) \exp(i\vec{q}\vec{u}) d\vec{u},$$

where u_i are fluctuating components of the instantaneous velocity. Coefficients of this expansion are termed cumulants (semi-invariants). They are more convenient to use as statistical characterisctics of distributions than the moments because of their invariant properties. The first four cumulants are:

$$C_1^i = \langle u_i \rangle, \quad C_2^{ij} = \langle u_i u_j \rangle, \quad C_3^{ijk} = \langle u_i u_j u_k \rangle,$$
$$C_4^{ijkl} = \langle u_i u_j u_k u_l \rangle - \langle u_i u_j \rangle \langle u_k u_l \rangle - \langle u_i u_k \rangle \langle u_j u_l \rangle - \langle u_i u_l \rangle \langle u_j u_k \rangle,$$

where u_i are fluctuating components of the instantaneous velocity. It follows from the foregoing that to describe the turbulent state of a flow it is sufficient to define cumulants of this expansion. As a rule, the infinite chain of transport equations for cumulants (moments) has to be closed by the application of various proposals (or hypotheses) for turbulence modeling, for example, the cumulant-free approach [1], [2] and others. Cumulants of the orders $1, \ldots, n$ are calculated from the differential transport equations. The $(n+1)$th-order cumulants, describing the processes of turbulent diffusion in the nth-order cumulants equations, are calculated from approximate algebraic expressions derived from the corresponding differential transport equations in which the $(n+2)$th-order cumulants are treated as zero. It should be noted that such an approximation can lead to incorrect results (negative values of quantities which in practice are strictly positive: PDF, dispersions, energy, dissipation). The last problem is a consequence of the fact that by defining a finite number of cumulants (moments) the corresponding probability may not even exists. It is closely connected to the principle impossibilty of arbitrarily breaking the Taylor series of the logarithm of the characteristic function after a finite number of terms.

In the present consideration the assumption of a relaxation character of the PDF evolution to the equilibrium state is applied. This assumption means that there will be no strong deviations of the PDF from its equilibrium state. It is well-known that the statistical structure of the turbulent energy spectrum in the inertial range is described with good accuracy by the equilibrium PDF.

In many cases, semiempirical closure models of turbulence, which include differential equations for the cumulants (moments) of hydrodynamical quantities [3], are among the basic methods of describing turbulence. The closure procedures for the cumulants (moments) often implicitly assume that the closed system of differential equations allow the existence of invariant sets (manifolds). As is pointed out in [4], closure relations are, as a rule, derived using empirical hypotheses and implying certain assumptions, which are often poorly justified.

The correctness of replacing a differential equation by the corresponding closure relation is verified by investigating the compatibility of the system of differential equations used in modeling of a turbulent flow with an added differential constraint (i.e. with the algebraic relation). In particular in the present context the method of differential constraints [5] can be used to justify algebraic turbulence models applied to the calculation of the cumulants (moments).

The notion of invariant manifolds introduced in [6] for an arbitrary system of evolution equations is a natural generalization of an invariant set of a system of ordinary differential equations and enables us to find certain classes of differential constraints. In [7] the method of local determining equations of a system of evolution equations was proposed which generalizes the defining equations of the symmetry groups and makes it possible to find invariant manifolds admissible by a system and then, in a final stage to obtain explicit solutions of the system. The method of differential constraints was used in [8], [9] for investigating the problem of the development of a shearless mixing layer using a third-order closure model. Due to the derived differential constraints, it was established that the equation of the invariant manifold (differential constraint of the model) coincides with the classical algebraic tensor-invariant Hanjalic–Launder model [10] for an unstratified flow and with the Zeman–Lumley model [11] for a stratified flow. Reduction of the model on the invariant manifolds made it possible to find self-similar solutions to the problem in case of an unstratified flow and to select a class of particular solutions [9].

Local-equilibrium approximations of second-order moments are invoked for modeling turbulent flows (see, e.g. [12]). In [13], these approximations were analyzed using the method of differential constraints. As an example, the dynamics of a far planar turbulent wake was investigated. It was established that the application of the local-equilibrium approximation is associated with a vanishing of the Poisson bracket for the defect of the averaged longitudinal velocity component U_1 and the turbulent energy e. Having the Poisson bracket equal to zero is the condition of compatibility of the overdetermined system consisting of the $(e, \epsilon, \langle u'v' \rangle)$ model of turbulence [15] and the algebraic model of local-equilibrium approximation for the tangential Reynolds stress $\langle u'v' \rangle$ [16], ϵ denotes the dissipation of the turbulence kinetic energy. Numerical experiments carried out in the far wake verify this result [13].

Therefore the method of differential constraints advanced by A. Cartan and N. Yanenko is of a special interest in view of its application to semiempirical models of turbulence.

In [14] the problem of shearless turbulent diffusion was studied by using Lie-group analysis where the authors found three different invariant solutions (scaling law): classical diffusion-like solution (heat equation like), decelerating diffusion-wave solution and finite domain diffusion due to rotation. It was proven that if only one spatial dimension is considered, models based on Reynolds averaging are only capable to model either the diffusion-like solution or the decelerating diffusion-wave solution. The latter solution is only admitted under certain algebraic constraints on the model constants. Turbulent diffusion on a finite domain induced by rotation is not admitted by any of the classical models.

The aim of the present article is to apply the method of differential constraints for formulating integrability properties of an infinite chain of transport equations for the cumulants which appear in modeling the dynamics of a momentumless turbulent planar wake. We expose the conditions of compatibility of the original infinite system of partial differential equations for cumulants with the gradient-type algebraic relation for the so-called triple correlations (the differential constraint) [10]. The compatibility conditions obtained make it possible to realize a reduction of the original infinite chain of transport equations for the cumulants and to present an algorithm for calculating cumulants of arbitrary order. This algorithm is based on a recursion relation. An illustrative example of applying the compatibility conditions obtained for examining a third-order closure model of turbulence is given.

The mathematical tools necessary for expression of the above-mentioned concept are provided by Symmetry Analysis [17]. We begin with a preliminary discussion and introduce basic notions.

2 General Notions

The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$u_t^i = \mathcal{F}^i(t, x_1, \ldots, x_n, u^1, \ldots, u_\lambda^k, \ldots,) \tag{2.1}$$

may be enlarged by appending additional differential equations (differential constraints)

$$h_j(t, x_1, \ldots, x_n, \ldots, u^1, \ldots, u^m, \ldots, u_\lambda^k, \ldots) = 0, \quad 1 \le j \le p \tag{2.2}$$

where $i = 1, \ldots, m$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $u_\lambda^k = \partial^\lambda u^k / \partial x_1^{\lambda_1} \ldots \partial x_n^{\lambda_n}$ such that the overdetermining system (2.1), (2.2) satisfies some conditions of compatibility. In practice, the classical methods for studying overdetermined systems may be difficult because it is necessary to find solutions to overdetermined systems using the Requeir-Ritt theory. The notion of invariant manifolds for a system of partial differential equations of a very general form (an extension of the invariant relations introduced by Levi-Chevita and Amaldy) allows us to find certain classes of differential constraints.

Below, we review certain definitions and statements from [6], [17], [18].

A dynamical system of ordinary differential equations

$$x_t^i = f^i(\vec{x}), \quad i = 1, \ldots, n, \quad \vec{x} = (x^1, \ldots, x^n)$$

generates the local one-parametric group G_1 with the vector field

$$V_f = f^1 \frac{\partial}{\partial x^1} + \cdots + f^n \frac{\partial}{\partial x^n}.$$

A regular manifold M given by the equations

$$g^1(\vec{x}) = \cdots = g^s(\vec{x}) = 0, \quad s < n,$$

is said to be the invariant manifold (set) under the group G_1 if

$$V_f(g^i)\big|_M = 0, \quad 1 \le i \le s.$$

Invariant manifolds (sets) of the above dynamical system are invariant manifolds of the corresponding one-parametric group G_1.

In full analogy to this definition we also define the invariant manifold for the system $\mathcal{F}$ of evolution equations (2.1).

Let $J^k(U, R^m)$ be a space of k-jets on $U \subset J^k(U, R^m)$. A manifold (set) $\mathcal{H} \subset J^k(R^{n+1}, R^m)$ given by equations (2.2) is said to be the invariant manifold (set) of system $\mathcal{F}$ if

$$V_{\mathcal{F}}(h_j)\big|_{[\mathcal{H}]_0} = 0,$$

$$V_{\mathcal{F}} = \frac{\partial}{\partial t} + \sum_{i=1}^{m} \mathcal{F}^i \frac{\partial}{\partial u^1} + \sum_{i=1}^{m} D^\alpha(\mathcal{F}^i) \frac{\partial}{\partial u^i_\alpha},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha = D^{\alpha_1}_{x_1} \ldots D^{\alpha_n}_{x_n}$ and $V_{\mathcal{F}}$ is a vector field generated by system $\mathcal{F}$.

The invariant condition can be written in the following equivalent form [6]

$$D_t(h_j)\big|_{[\mathcal{F}]_0}\big|_{[\mathcal{H}]_0} = 0. \tag{2.3}$$

Here $[\mathcal{F}]_0$ is the ∞-prolongation (see [6]) of $\mathcal{F}$ with respect to $x_1, \ldots, x_n$. The set $[\mathcal{H}]_0$ is determined by analogy.

It should be noted that the existence of an invariant manifold of the system $\mathcal{F}$ and a solution to the overdetermined system (2.1), (2.2) are connected with each other. If the system $\mathcal{F}$ has an invariant manifold of the form $\mathcal{H}$, then under a suitable condition on the smoothness of the functions h_i there exists a unique solution to the system (2.1), (2.2). It follows from the fact that the conditions of compatibility and invariance are equivalent (see [6] for more details). Moreover, symmetries of differential equations can generate invariant manifolds. Indeed, in the case of system $\mathcal{F}$ symmetries satisfy the so-called defining equations

$$D_t(h_j) = \sum_{k=1}^{p} a^{ik}_n D^n_k(h_k) + \cdots + a^{ik}_0 h_k, \quad 1 \le j \le p, \tag{2.4}$$

where a^{ik}_j depends on x, t, functions u^l and derivatives u^l_λ. The defining equations must be satisfied by virtue of the system $\mathcal{F}$. If h_j satisfy the defining equations, then the following equalities

$$D_t(h_j)\big|_{[\mathcal{F}]_0}\big|_{[\mathcal{H}]_0} = 0$$

are satisfied and coincide with the condition of invariance of the set $\mathcal{H}$ with respect to the system $\mathcal{F}$.

For the current purpose we also consider a scalar evolution equation which can be written in the form

$$u_t = F(t, x, u, u_1, \ldots, u_\lambda, \ldots). \tag{2.5}$$

Symmetries of the form $h = h(t, x, u, u_1)$ are called contact symmetries. At this point we note that in the classical case the term "symmetry" means that for any solution $h = h(t, x, u, u_1)$ of the defining equation we can find an one-parametric group of the transformations such that equation (2.5) does not change the form written in the new variables $\tilde{t}$, $\tilde{x}$, $\tilde{u}$, $\tilde{u}_\lambda$. Such group represents the translation group along characteristics of the corresponding equation. In the general case, we introduce the Lie bracket

$$[f, g] = g_*(f) - f_*(g), \tag{2.6}$$

for two functions $f = f(t, x, u, u_1, \dots)$ and $g = g(t, x, u, u_1, \dots)$. The symbols f_* and g_* denote the differential operator

$$\sum_i \partial_i f D^i, \quad \partial_i = \frac{\partial}{\partial u_i}.$$

In terms of Lie brackets (2.6), the symmetry by definition is a function commuting with the right-hand side of equation (2.5). The set of all symmetries of equation (2.5) is the Lie algebra L and the set of all contact symmetries is the subalgebra Lie S in L such that $dimS \le n + 3$ where n is the order of equation (2.5) [18]. Furthermore, every m-dimensional subalgebra $\mathcal{O} \subset S$ defines the so-called differential substitution

$$\tilde{x} = \varphi(x, u, \dots, u_m), \quad \tilde{u} = \psi(x, u, \dots, u_m), \quad |\partial_m \varphi| + |\partial_m \psi| \neq 0$$

which connects equation (2.5) with an equation of the form

$$\tilde{u}_t = \Phi(t, \tilde{x}, \tilde{u}, \dots, \tilde{u}_n).$$

For example, it is easy to check that the differential substitution $\varphi = x$, $\psi = u_1/u$ transforms solutions $u(x, t)$ of the heat equation into solutions $\tilde{u}(\tilde{x}, t)$ of the Burgers equation. It should be noted that the differential substitutions also appear in the context of finding compatible differential constraints for systems of partial differential equations which appear in semiempirical models of turbulence. For example, the differential constraint obtained for the third-order closure model for a shearless turbulent flow has the form of differential substitution, see [8]. It is interesting to note this differential substitution realizes the so-called algebraic triple correlation model by Zeman and Lumley [11].

3 The governing equations

Momentumless turbulent wakes behind bodies were considered in a number of works. It has been shown (see, for example [19]) that the momentumless turbulent wakes behave quite differently from the turbulent wakes with a nonzero excess impulse. In particular, we may account that the momentumless turbulent wake is also a shearless turbulent flow. There exists a sufficiently large number of publications [19]–[23] (comprehensive references can be found therein) in which the results of experimental and theoretical investigations are discussed. The results of laboratory and numerical experiments demonstrate a faster decrease of the defect of the longitudinal velocity component in comparison with a wake flow behind a towed body. As a result, a turbulent wake, which is considered on a distance about ten diameters behind a body, is a shearless flow in practice. The latter implies that the velocity of a flow coincides with the incident stream velocity, while the tangential Reynolds stresses are equal to zero.

In order to describe the flow in a far planar (momentumless) turbulent wake in a non-stratified fluid we consider the following family of transport equations for cumulants [24] ($n = 2, \dots, \infty$)

$$U_0 \frac{\partial C_n}{\partial x} + \frac{\partial C_{n+1}}{\partial y} + A_1^n C_1 \frac{\partial C_n}{\partial y} + A_2^n C_2 \frac{\partial C_{n-1}}{\partial y} + A_3^n C_3 \frac{\partial C_{n-2}}{\partial y} + \dots + A_n^n C_n \frac{\partial C_1}{\partial y} + B_n \frac{C_n}{\tau} = 0, \tag{3.7}$$

where $\tau = e/\epsilon$, $e = 3/2\langle u_2^2 \rangle$, $C_n \equiv C_n^{2\dots2}$ are the constants. Here, e and τ are the kinetic energy and the time scale of turbulence respectively, ϵ is the rate of dissipation, $\langle u_2^2 \rangle$ denotes the one-point velocity correlation of the second-order. U_0 is the remote velocity. The constant involved in the system are denoted by A_n^n, B_n where n is upper and lower index. Introducing the new variable $t = \Theta(x) \equiv U_0^{-1} x$, equation (3.7) can be rewritten (using the new coordinates (y, t)) in the form of evolution equation

$$\frac{\partial \bar{C}_n}{\partial t} + \frac{\partial \bar{C}_{n+1}}{\partial y} + A_1^n \bar{C}_1 \frac{\partial \bar{C}_n}{\partial y} + A_2^n \bar{C}_2 \frac{\partial \bar{C}_{n-1}}{\partial y} + A_3^n \bar{C}_3 \frac{\partial \bar{C}_{n-2}}{\partial y} + \dots + A_n^n \bar{C}_n \frac{\partial \bar{C}_1}{\partial y} + \frac{2}{3} B_n \frac{\bar{C}_n}{\bar{\tau}} = 0, \tag{3.8}$$

where $\bar{e} = e(\Theta^{-1}(t), y)$, $\bar{\epsilon} = \epsilon(\Theta^{-1}(t), y)$, $\bar{C}_n = C_n(\Theta^{-1}(t), y)$ and $\bar{\tau} = \bar{C}_2/\bar{\epsilon}$. We complete the system by the $\bar{\epsilon}$-equation

$$\frac{\partial \bar{\epsilon}}{\partial t} = \frac{\partial}{\partial y} \left[\bar{c}_d \bar{\tau} \bar{C}_2 \frac{\partial \bar{\epsilon}}{\partial y} \right] - \bar{c}_{\epsilon_2} \frac{\bar{\epsilon}}{\bar{\tau}}, \tag{3.9}$$

where $\bar{c}_d = c_\mu/\sigma_\epsilon$, and $\bar{c}_{\epsilon_2} = 2c_{\epsilon_2}(c_1 + 2)/3c_1$. The above mathematical model is based on the assumption of a relaxation character of the PDF evolution to an equilibrium state.

In order to study the mathematical model under consideration, we first rewrite (3.8) in the form of a system of evolution equations where the quantities A_n^n and B_n for $n = 2, 3$ are specified by empirical constants c_1, c_2, c_s, c_μ, σ_ϵ and c_{ϵ_2} taken from [15]. Equation (3.8) thus provides the following evolution nonclosed system for the cumulants (of course this system can be closed by appending the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ using equation (3.8))

$$\frac{\partial \bar{C}_2}{\partial t} = -\frac{\partial \bar{C}_3}{\partial y} - \alpha \frac{\bar{C}_2}{\bar{\tau}}, \tag{3.10}$$

$$\frac{\partial \bar{C}_3}{\partial t} = -\frac{\partial C_4}{\partial y} - 2A_2^3 \bar{C}_2 \frac{\partial \bar{C}_2}{\partial y} - \gamma \frac{\bar{C}_3}{\bar{\tau}}, \tag{3.11}$$

extended by (3.9), where $\alpha = 2/3$, $\gamma = 2c_2(c_1 + 1)/3c_1$, $\bar{c}_{\epsilon_2} = 2c_{\epsilon_2}(c_1 + 2)/3c_1$, $\bar{c}_d = c_\mu/\sigma_\epsilon$, the value of A_2^3 will be specified below. The terms containing $\bar{C}_1$ cancel in view of the physical setting of the original problem.

4 Compatible differential constraints and reduction

Subsequently we investigate the system of evolution equations (3.9)–(3.11) (completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$) from the point of view of the reduction theory of integrable systems. In particular, the notion of compatible differential constraints was used in [25], [26] to find some classes of reductions of the normalized and Schwarzian hierarchy of infinite-dimensional systems of hydrodynamic type.

The aim of this section is to find a reduction of system (3.9)–(3.11) (completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$) defined by a differential constraint of the form

$$\bar{C}_3 = -3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_2}{\partial y}. \qquad (4.12)$$

From a physical point of view, the way of derivation of an algebraic relationship of this type for the triple correlations was presented by Hanjalic and Launder in the cases of axisymmetric and plane jets [10]. At first, by appending the differential constraint (4.12) to the system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ we find conditions of compatibility of the overdetermined system obtained. Then, we show how the conditions obtained enable us to make a reduction of the original infinite system of evolution equations. In Theorem 4.1 we find the conditions of compatibility of the overdetermined system under consideration. Then we show (see, Theorem 4.2) that conditions (4.13) are fulfilled in view of the equation for $\bar{\tau}$.

Theorem 4.1 *Let the function $\bar{\tau} = \bar{C}_2/\bar{\epsilon}$ satisfy the relations*

$$\frac{\partial \bar{\tau}}{\partial y} = 0, \quad \frac{\partial \bar{\tau}}{\partial t} = 2\alpha. \qquad (4.13)$$

Assume that $2A_2^3 = 3\bar{c}_s\gamma$. Then the overdetermined system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ and the differential constraint (4.12) is compatible if and only if

$$\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_3}{\partial y} \qquad (4.14)$$

.

The proof is based on direct calculations. Indeed, let us assume that the system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ and (4.12) is compatible. Calculating the time derivative for the function $\bar{C}_3 = -3\bar{c}_s\bar{\tau}\bar{C}_2\partial\bar{C}_2/\partial y$, we obtain that

$$\frac{\partial \bar{C}_3}{\partial t} = -\left[3\bar{c}_s\bar{C}_2\frac{\partial \bar{\tau}}{\partial t}\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\bar{\tau}\frac{\partial \bar{C}_2}{\partial t}\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial}{\partial t}\frac{\partial \bar{C}_2}{\partial y}\right].$$

Using equation (3.10), we can rewrite this formula as follows

$$\frac{\partial \bar{C}_3}{\partial t} = -\left[3\bar{c}_s\bar{C}_2\frac{\partial \bar{\tau}}{\partial t}\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\bar{\tau}\left(-\frac{\partial \bar{C}_3}{\partial y} - \alpha\frac{\bar{C}_2}{\bar{\tau}}\right)\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\bar{C}_2\bar{\tau}\frac{\partial}{\partial y}\left(-\frac{\partial \bar{C}_3}{\partial y} - \alpha\frac{\bar{C}_2}{\bar{\tau}}\right)\right].$$

Taking into account that $\partial\bar{\tau}/\partial y = 0$ this formula can be written in the form

$$\frac{\partial \bar{C}_3}{\partial t} = -\left[3\bar{c}_s\bar{C}_2\frac{\partial \bar{C}_2}{\partial y}\left(\frac{\partial \bar{\tau}}{\partial t} - 2\alpha\right) - 3\bar{c}_s\bar{\tau}\frac{\partial}{\partial y}\left(\bar{C}_2\frac{\partial \bar{C}_3}{\partial y}\right)\right].$$

Replacing the derivative $\partial \bar{C}_3/\partial t$ and the function $\bar{C}_3$ in equation (3.11) by their representation from the above formula and (4.12), we obtain (using the assumption (4.13) and the equality $2A_2^3 = 3\bar{c}_s\gamma$) that

$$\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_3}{\partial y}.$$

In the following we assume that $\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\partial\bar{C}_3/\partial y$. Our aim is to prove that the overdetermined system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ and the differential constraint (4.12) is compatible. To establish this property, it is sufficient to validate that the set given on the space 1-jets

$$\mathcal{D} = \{\bar{C}_2, \partial\bar{C}_2/\partial y, \bar{C}_3, \bar{\tau} \,:\, \mathcal{H}^1(\bar{C}_2, \partial\bar{C}_2/\partial y, \bar{C}_3, \bar{\tau}) \equiv \bar{C}_3 + 3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_2}{\partial y} = 0\} \tag{4.15}$$

is an invariant set under the flow generated by the system (3.9)–(3.11). To this end, it is sufficiently to check the invariant criterion (2.3) on solutions of the original system. Calculating the time derivative, we obtain

$$D_t(\mathcal{H}^1) = \frac{\partial \bar{C}_3}{\partial t} + 3\bar{c}_s\frac{\partial}{\partial t}\left[\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_2}{\partial y}\right]. \tag{4.16}$$

Using equation (3.11) and formula (4.14), we can rewrite (4.16) as follows

$$D_t(\mathcal{H}^1) = \frac{\partial}{\partial y}\left[3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial \bar{C}_3}{\partial y}\right] - 2A_2^3\bar{C}_2\frac{\partial \bar{C}_2}{\partial y} - \gamma\frac{\bar{C}_3}{\bar{\tau}}$$

$$+3\bar{c}_s\frac{\partial \bar{\tau}}{\partial t}\frac{\partial \bar{C}_2}{\partial y}\bar{C}_2 + 3\bar{c}_s\bar{\tau}\frac{\partial \bar{C}_2}{\partial t}\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial^2 \bar{C}_2}{\partial t\partial y}.$$

Replacing the derivatives $\partial\bar{C}_2/\partial t$ and $\partial^2\bar{C}_2/\partial t\partial y$ by their representations from equation (3.10), we obtain

$$D_t(\mathcal{H}^1) = 3\bar{c}_s\bar{\tau}\frac{\partial \bar{C}_2}{\partial y}\frac{\partial \bar{C}_3}{\partial y} + 3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial^2 \bar{C}_3}{\partial y^2} - 2A_2^3\bar{C}_2\frac{\partial \bar{C}_2}{\partial y} - \gamma\frac{\bar{C}_3}{\bar{\tau}}$$

$$+3\bar{c}_s\frac{\partial \bar{\tau}}{\partial t}\frac{\partial \bar{C}_2}{\partial y}\bar{C}_2 + 3\bar{c}_s\bar{\tau}\left[-\frac{\partial \bar{C}_3}{\partial y} - \alpha\frac{\bar{C}_2}{\bar{\tau}}\right]\frac{\partial \bar{C}_2}{\partial y}$$

$$+3\bar{c}_s\bar{\tau}\bar{C}_2\left[-\frac{\partial^2 \bar{C}_3}{\partial y^2} - \frac{\alpha}{\bar{\tau}}\frac{\partial \bar{C}_2}{\partial y}\right].$$

We can rewrite this formula in the form

$$D_t(\mathcal{H}^1) = 3\bar{c}_s\bar{\tau}\frac{\partial \bar{C}_2}{\partial y}\frac{\partial \bar{C}_3}{\partial y} + 3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial^2 \bar{C}_3}{\partial y^2} - 2A_2^3\bar{C}_2\frac{\partial \bar{C}_2}{\partial y}$$

$$-\frac{\gamma}{\bar{\tau}}\mathcal{H}^1 + 3\gamma\bar{c}_s\bar{C}_2\frac{\partial \bar{C}_2}{\partial y} + 3\bar{c}_s\frac{\partial \bar{\tau}}{\partial t}\frac{\partial \bar{C}_2}{\partial y}\bar{C}_2$$

$$-3\bar{c}_s\bar{\tau}\frac{\partial\bar{C}_3}{\partial y}\frac{\partial\bar{C}_2}{\partial y}-6\bar{c}_s\alpha\frac{\partial\bar{C}_2}{\partial y}\bar{C}_2-3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial^2\bar{C}_3}{\partial y^2}.$$

Using the assumption that $2A_2^3 = 3\bar{c}_s\gamma$, we obtain

$$D_t(\mathcal{H}^1) = 3\bar{c}_s\frac{\partial\bar{C}_2}{\partial y}\bar{C}_2\left(\frac{\partial\bar{\tau}}{\partial t}-2\alpha\right)-\frac{\gamma}{\bar{\tau}}\mathcal{H}^1. \tag{4.17}$$

It follows from (4.12) that

$$D_t(\mathcal{H}^1) = -\frac{\gamma}{\bar{\tau}}\mathcal{H}^1.$$

Therefore

$$D_t(\mathcal{H}^1)\big|_{\mathcal{H}^1=0} = 0.$$

This completes the proof of the theorem.

Remark 4.1 The algebraic model (4.12) appears in the context of applying the so-called local-algebraic approximation to equation (3.11) where $2A_2^3 = 3\bar{c}_s\gamma$ (see, e.g. [12]).

Theorem 4.1 allows us to find a reduction of system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$. The equation for $\bar{\tau}$ will be crucial for our study of the invariant manifolds of this system. This equation can be obtained from (3.9), (3.10). Indeed, calculating the time derivative for $\bar{\tau}$ and using equations (3.9) and (3.10), we have

$$\begin{aligned}\frac{\partial\bar{\tau}}{\partial t} = -\frac{\bar{\tau}}{\bar{C}_3}\left[\frac{\partial\bar{C}_3}{\partial y}+\bar{c}_d\bar{\tau}\bar{C}_2\frac{\partial^2\bar{C}_2}{\partial y^2} \right. &+ \left.\bar{c}_d\bar{\tau}\left(\frac{\partial\bar{C}_2}{\partial y}\right)^2\right]\\ &+ \bar{c}_d\bar{C}_2\bar{\tau}\frac{\partial^2\bar{\tau}}{\partial y^2}+2\bar{c}_d\bar{\tau}\frac{\partial\bar{\tau}}{\partial y}\frac{\partial\bar{C}_2}{\partial y}-\bar{c}_d\left(\frac{\partial\bar{\tau}}{\partial y}\right)^2\bar{C}_2+\bar{c}_{\epsilon_2}-\alpha.\end{aligned}$$

Theorem 4.2 *Let $\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\partial\bar{C}_3/\partial y$ and $2A_2^3 = 3\bar{c}_s\gamma$. Assume that $\bar{c}_d = 3\bar{c}_s$ and $3\alpha = \bar{c}_{\epsilon_2}$. Then system (3.9)–(3.11) completed by the transport equations for cumulants $\bar{C}_n$, $n \geq 4$ admits the invariant manifold $\mathcal{D}$ and its reduction on the set $\mathcal{D}$ is of the form*

$$\frac{\partial\bar{\epsilon}}{\partial t} = \frac{\partial}{\partial y}\left[\bar{c}_d\bar{\tau}\bar{C}_2\frac{\partial\bar{\epsilon}}{\partial y}\right]-\bar{c}_{\epsilon_2}\frac{\bar{\epsilon}}{\bar{\tau}}, \tag{4.18}$$

$$\bar{C}_2 = \bar{\tau}\bar{\epsilon}, \tag{4.19}$$

$$\bar{C}_3 = -3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial\bar{C}_2}{\partial y}, \tag{4.20}$$

$$\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\frac{\partial\bar{C}_3}{\partial y} \tag{4.21}$$

$$\begin{aligned}\frac{\partial\bar{C}_n}{\partial t} &+ \frac{\partial\bar{C}_{n+1}}{\partial y}+A_2^nC_2\frac{\partial\bar{C}_{n-1}}{\partial y}+A_3^nC_3\frac{\partial\bar{C}_{n-2}}{\partial y}+\dots\\ &+ A_{n-1}^n\bar{C}_{n-1}\frac{\partial\bar{C}_2}{\partial y}+\frac{2}{3}B_n\frac{\bar{C}_n}{\bar{\tau}}=0, \quad n\geq 4\end{aligned} \tag{4.22}$$

where the function $\bar{\tau}(y,t) \equiv \bar{\tau}(t)$ solves the ordinary differential equation

$$\frac{d\bar{\tau}}{dt} = \bar{c}_{\epsilon_2}-\alpha \qquad (\textit{a version of equation for } \bar{\tau} \textit{ on the set } \mathcal{D}). \tag{4.23}$$

To show that $\mathcal{D}$ is an invariant manifold of system (3.9)–(3.11), it is sufficient to check that the equation for $\bar{\tau}$ admits a class of solutions which satisfy conditions (4.13). To find this class of solutions, we consider the equation for $\bar{\tau}$. It is clear that this equation on the set $\mathcal{D}$ has a class of exact solutions of the form $\bar{\tau}(y,t) \equiv \bar{\tau}(t) = (\bar{c}_{\epsilon_2} - \alpha)t + const$ (taking into account that for $\bar{c}_d = 3\bar{c}_s$ the expression in the square brackets equals zero due to the definition of the set $\mathcal{D}$, see the equation for $\bar{\tau}$). Using the equality $3\alpha = \bar{c}_{\epsilon_2}$, we conclude that conditions (4.13) of Theorem 4.1 are realized.

Therefore the differential constraint $\bar{C}_4 = -3\bar{c}_s\bar{\tau}\bar{C}_2\partial\bar{C}_3/\partial y$ determines the integrable hierarchy (4.18)–(4.22) associated with the original system (3.8),(3.9) in the following sense. The equation

$$\frac{\partial \bar{\epsilon}}{\partial t} = \frac{\partial}{\partial y}\left[\bar{c}_d\bar{\tau}\bar{C}_2\frac{\partial \bar{\epsilon}}{\partial y}\right] - \bar{c}_{\epsilon 2}\frac{\bar{\epsilon}}{\bar{\tau}},$$

(where $\bar{C}_2 = \bar{\tau}\bar{\epsilon}$) appears in the context of the theory of the Porous Medium Equation (see, [27]). In particular, it is known that the initial (boundary) value problems for equations of this type can be solved in a class of nonnegative weak solutions; solutions are C^∞-smooth in their positivity set where $\bar{\epsilon} > 0$; there exists an interface or free boundary separating regions where $\bar{\epsilon} > 0$ from regions where $\bar{\epsilon} = 0$. Therefore (4.18) is one of the canonical integrable equations. Indeed, if we set

$$\theta \equiv \theta(t) = \int_0^t \bar{\tau}^2(p)dp, \quad \varsigma(\theta) = \bar{\tau}(\theta^{-1}(t)), \quad \psi(\theta) = \frac{1}{\varsigma^3(\theta)}$$

and

$$\bar{\epsilon}(y,\theta) = u(y,\theta)\exp(-\int_0^\theta \psi(p)dp), \qquad \text{where } \bar{\epsilon}(z,\theta) = \bar{\epsilon}(z,t)$$

($\theta(t)$ maps $[0,+\infty)$ onto $[0,+\infty)$), then for u we have:

$$\frac{\partial u}{\partial \bar{\theta}} = \frac{\partial}{\partial y}\left[3\bar{c}_s u\frac{\partial u}{\partial y}\right], \qquad \text{where } \bar{\theta} = \int_0^\theta \exp(-\int_0^\xi \psi(p)dp)d\xi, \tag{4.24}$$

It is easy to check that $\bar{\theta} : [0,+\infty) \to [0,+\infty)$. Thus, under the change of variables we obtained the well-known equation (4.24) which was proposed by Boussinesq for modeling the flow of compressible fluids through a porous medium [28]. As a result, we can find various classes of solutions to (4.24), for example, the Barenblatt–type solutions [29]. Thus we derive at the following: the formulas (4.19), (4.20) and (4.21) determine the cumulants $\bar{C}_2$, $\bar{C}_3$ and $\bar{C}_4$ for a given function $\bar{\epsilon}$. The cumulants $\bar{C}_{n+1}$, $n \geq 4$ can be found by the following recursion relation (see, formula (4.22))

$$\frac{\partial \bar{C}_{n+1}}{\partial y} = -\left(\frac{\partial \bar{C}_n}{\partial t} + A_2^n C_2\frac{\partial \bar{C}_{n-1}}{\partial y} + A_3^n C_3\frac{\partial \bar{C}_{n-2}}{\partial y} + \cdots + A_{n-1}^n\bar{C}_{n-1}\frac{\partial \bar{C}_2}{\partial y} + \frac{2}{3}B_n\frac{\bar{C}_n}{\bar{\tau}}\right). \tag{4.25}$$

The boundary conditions for $\bar{C}_n$ are defined by the physical model, we have $\bar{C}_n \equiv 0$ outside of turbulent wake. Thus, system (3.9)–(3.9) admits a class of solutions such that every subsequent cumulant $\bar{C}_{n+1}$ can be presented by preceding cumulants $\bar{C}_i$, $i = 2, \dots n$ in the form of corresponding differential substitutions. As it was indicated in section 2, this property means that system (3.9)–(3.11) admits invariants under the flow generated by the the original system which correspond to certain subalgebras of the Lie algebra of classical symmetries (see, [18] for more details).

Theorem 4.2 allows us to extract a finite closed system of differential equations of the following form

$$\frac{\partial \bar\epsilon}{\partial t} = \frac{\partial}{\partial y}\left[3\bar c_s \bar\tau \bar C_2 \frac{\partial \bar\epsilon}{\partial y}\right] - \bar c_{\epsilon 2}\frac{\bar\epsilon}{\bar\tau}, \tag{4.26}$$

$$\frac{\partial \bar C_2}{\partial t} = -\frac{\partial \bar C_3}{\partial y} - \alpha\frac{\bar C_2}{\bar\tau}, \tag{4.27}$$

$$\frac{\partial \bar C_3}{\partial t} = \frac{\partial}{\partial y}\left[3\bar c_s \bar\tau \bar C_2 \frac{\partial \bar C_3}{\partial y}\right] - 3\bar c_s \gamma \bar C_2 \frac{\partial \bar C_2}{\partial y} - \gamma\frac{\bar C_3}{\bar\tau} \tag{4.28}$$

such that equations (4.18)–(4.20) determine its reduction on the invariant manifold. The above system coincides with a "standard" third-order closure model of turbulence. As a remark, we note that the compatible condition

$$\bar C_4 = -3\bar c_s \bar\tau \bar C_2 \frac{\partial \bar C_3}{\partial y}$$

for the system (3.9)–(3.11), (4.12) also coincides with the algebraic relation for the fourth-order cumulant $\bar C_4$ which is employed in numerical simulations [2]. The following example illustrates how the reduction above enables use to construct an explicit solution to system (4.26)–(4.28).

System (4.26)–(4.28) admits a two-parametric group of scale transformation. Therefore, we can find a selfsimilar solution of the form

$$\bar C_2 = \frac{f(\xi)}{(t+t_0)^{2\mu}}, \quad \bar C_3 = \frac{g(\xi)}{(t+t_0)^{3\mu}}, \quad \bar\epsilon = \frac{h(\xi)}{(t+t_0)^{3\mu+\nu}}, \quad \xi = \frac{y}{L}, \quad L = (t+t_0)^{\nu}. \tag{4.29}$$

If we choose $\nu = 1-\mu$, then the original system is transformed to the system of ordinary differential equations for the profiles f, g and h:

$$3\bar c_s\left(\frac{f^2}{h}h_\xi\right)_\xi + (1-\mu)\xi h_\xi + (2\mu+1)h - \bar c_{\epsilon_2}\frac{h^2}{f} = 0, \tag{4.30}$$

$$3\bar c_s\left(\frac{f^2}{h}g_\xi\right)_\xi - 3\bar c_s f f_\xi + (1-\mu)\xi g_\xi - \gamma\frac{gh}{f} + 3\mu g = 0, \tag{4.31}$$

$$2\mu f + (1-\mu)\xi f_\xi - g_\xi - \alpha h = 0. \tag{4.32}$$

The free similarity exponent μ has to be determined from a solution of the nonlinear eigenvalue problem. This is a typical situation appearing in nonlinear diffusion problems where a conservation law does not exist. To find a solution to system (4.30)–(4.32), we use the existence of the invariant manifolds obtained in Theorem 4.2. As a result, we have the following boundary value problem for $h(\xi)$ (setting $\mu = \bar c_{\epsilon_2}/2(\bar c_{\epsilon_2} - \alpha)$):

$$(\delta w_c^2 h h_\xi + (1-\mu)\xi h)_\xi = 0, \quad w_c = \bar c_{\epsilon_2} - \alpha, \tag{4.33}$$

$$h(-\infty) = 0, \quad h(+\infty) = 0, \quad \delta = const. \tag{4.34}$$

We note that equation (4.33) arises in the context of studying a selfsimilar solution to the porous medium equation, see [29]. Therefore there exists the unique solution $h(\xi)$ to problem (4.33)–(4.34) which coincides with the well-known Barenblatt solution [29]. The functions $f(\xi)$ and $g(\xi)$ are determined by the formulas $f(\xi) = (\bar{c}_{\epsilon_2} - \alpha)h(\xi)$ and $g(\xi) = -3\bar{c}_s(\bar{c}_{\epsilon_2} - \alpha)f(\xi)f_\xi(\xi)$.

Remark 4.2 The values $\mu = 3/4$ and $\nu = 1/4$ agree well with Kolmogorov's law of decaying isotropic turbulent flow (to calculate the constant μ, it is used the above presented formula for μ). These quantities are also close to the values suggested in [22].

5 Conclusions

We showed that the method of differential constraints is an effective tool to analyze parametric turbulent models for planar momentumless wakes that enables us, in particular, to find new reductions of the model under consideration, and then in turn to construct explicit solutions. An equation of the form (4.18) which appears in the reduced chain can be analyzed by the standard Lie group method for finding group-invariant solutions (see, e.g. [30]). An important application of the approach presented is obtaining functional and algebraic relationships between various flow characteristics in exact analytic form. Moreover, it is turned out that some empirical model constants may be calculated and their values obtained are sufficiently close to the standard data.

An obvious reduction of (3.8) arises when only a finite number of cumulants in the infinite chain (3.8) are different from zero, see [1]. Millionshchikov's quasinormality hypothesis for the parametrization of diffusion processes in equations for triple correlation is an particular case of such reduction. Recall that according to this hypothesis, all cumulants of fourth- and higher-order are negligibly small in comparison with the corresponding correlation functions. Millionshchikov's quasinormality hypothesis is known to be defective in some cases that leads to physically contradictory results, see [3]. The advantage of our approach is based on finding invariant manifolds of an infinite chain of partial differential equations in order to obtain algebraic expressions for the n-order cumulants (moments).

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