Iterated Strict Dominance in General Games

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This Version: July 2005

Abstract

Following Milgrom and Roberts [*Econometrica* **58**(1990), 1255–1278], we offer a definition of iterated elimination of strictly dominated strategies (IESDS*) for games with (in)finite players, (non)compact strategy sets, and (dis)continuous payoff functions. IESDS* is always a well-defined order independent procedure that can be used to solve out Nash equilibrium in dominance-solvable games. We characterize IESDS* by means of a "stability" criterion. We show by an example that IESDS* might generate spurious Nash equilibria in the class of Reny's better-reply secure games. We provide sufficient conditions under which IESDS* preserves the set of Nash equilibria. *JEL Classification*: C70, C72.

Keywords: Game theory, strict dominance, iterated elimination, Nash equilibrium, Reny's better-reply secure games, well-ordering principle

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1 Introduction

Iterated strict dominance is perhaps one of the most basic principles in game theory. The concept of iterated strict dominance rests on the following simple idea: no player would play strategies for which some alternative strategy can yield him/her a greater payoff regardless of what the other players play and this fact is common knowledge. This concept has been used to expound the fundamental conflict between individual and collective rationality as illustrated by the Prisoner's Dilemma, and is closely related to the global stability of the Cournot-tatonnement process in terms of dominance solvability of games (cf. Moulin 1984; Milgrom and Roberts 1990). In particular, it has fruitful applications in Carlsson and van Damme's (1993) global games (see Morris and Shin 2003 for a survey).

A variety of elimination procedures has been studied by game theorists.[1] Among the most interesting questions that have been explored are: Does the order of elimination matter? Is it possible that the iterated elimination process fails to converge to a maximal reduction of a game? What are the sufficient conditions for existence and uniqueness of maximal reduction? Can a maximal reduction generate spurious Nash equilibria?

In the most general setting (where the number of players can be infinite, strategy sets can be in general topological spaces, and payoff functions can be discontinuous) Dufwenberg and Stegeman (2002) (henceforth DS) investigated the properties of a definition of *iterated elimination of (strictly) dom-*

[1]See in particular Moulin (1984), Gilboa, Kalai, and Zemel (1990), Stegeman (1990), Milgrom and Roberts (1990), Borgers (1993), Lipman (1994), Osborne and Rubinstein (1994), among others. (See also Jackson (1992) and Marx and Swinkels (1997) for iterated weak dominance.)

inated strategies (IESDS). Among others, DS demonstrated that (i) IESDS is in general an order dependent procedure, (ii) a maximal reduction may fail to exist, and (iii) IESDS can generate spurious Nash equilibria even in "dominance-solvable" games.[2] As DS pointed out, these anomalies and pathologies appear to be rather surprising and somewhat counterintuitive. DS (2002, p. 2022) concluded that:

> The proper definition and role of iterated strict dominance is unclear for games that are not compact and continuous. ... The identification of general classes of games for which IESDS is an attractive procedure, outside of the compact and continuous class, remains an open problem.

The main purpose of this paper is to offer a definition of IESDS that is suitable for all games, possibly with an arbitrary number of players, arbitrary strategy sets, and arbitrary payoff functions. This definition of IESDS will be denoted by IESDS* (the asterisk $*$ is used to distinguish it from other forms of IESDS). We will show that IESDS* is a well-defined order independent procedure: it yields a unique maximal reduction (see Theorem 1). This nice property is completely topology-free. For games that are compact and continuous, our IESDS* yields the same maximal reduction as DS's definition of IESDS (see Theorem 2). We also provide a characterization of IESDS* in terms of a "stability" criterion (see Theorem 3).

The IESDS* proposed in this paper is based mainly upon Milgrom and Roberts' (1990, pp. 1264-1265) definition of IESDS in a general class of su-

[2]DS also provided sufficient conditions for positive results. In particular, if strategy spaces are compact Hausdorff spaces and payoff functions are continuous, then DS's definition of IESDS yields a unique maximal reduction.

permodular games,[3] and has two major features: (1) IESDS* allows for an uncountable number of rounds of elimination, and is thus more general than DS's IESDS procedure, and (2) in each round of elimination, IESDS* allows for eliminating dominated strategies (possibly by using strategies that have previously been eliminated), rather than eliminating *only* those strategies that are dominated by some uneliminated strategy. These two features endow the IESDS* procedure with greater elimination power than DS's IESDS procedure.

The rationale behind the two features of IESDS* is as follows. Recall that a prominent justification for IESDS is "common knowledge of rationality"; see, e.g., Bernheim (1984), Osborne and Rubinstein (1994, Chapter 4), Pearce (1984), and Tan and Werlang (1988). While the equivalence between IESDS and the strategic implication of "common knowledge of rationality" has been established for games with compact strategy spaces and continuous payoff functions (see Bernheim 1984, Proposition 3.1), Lipman (1994) demonstrated that, for a more general class of games, there is a non-equivalence between *countably infinite iterated elimination of never-best replies* and "common knowledge of rationality". In particular, he showed that the equivalence can be restored by "removing never best replies as often as necessary" (p. 122), i.e., by allowing for an uncountably infinite iterated elimination of never-best replies (see Lipman 1994, Theorem 2). Therefore, it seems fairly natural and desirable to define IESDS for general games by allowing for an

[3] Ritzberger (2002, Section 5.1) considered a similar definition of IESDS for compact and continuous games that allows for eliminating strategies that are dominated by an uneliminated or eliminated strategy. We are grateful to Martin Dufwenberg for drawing our attention to this. (See also Brandenburger, Friedenberg, and Keisler's (2004) an analogous Definition 3.3 which allows for dominance by mixed strategy.)

uncountably infinite iterated elimination. Example 1 in Section 2 shows that IESDS* requires an uncountably infinite number of rounds to converge to a maximal reduction.

The second feature of IESDS* is in the same spirit as Milgrom and Roberts' (1990, pp. 1264-1265) definition of IESDS.[4] That is, in each round of elimination, IESDS* allows for eliminating dominated strategies, rather than eliminating *only* those strategies that are dominated by some uneliminated strategy in that round. For games where strategy spaces are compact and payoff functions are uppersemicontinuous in own strategies, this feature does not imply giving IESDS* more elimination power than DS's IESDS procedure, because it can be shown that, in this class of games, for any dominated strategy, there is some remaining uneliminated strategy that dominates it (see DS's Lemma, p. 2012).[5] However, for more general games, the second feature of IESDS* gives it more elimination power than DS's IESDS procedure.

To see this, consider a simple one-person game where the strategy space is $(0, 1)$ and the payoff function is $u(x) = x$ for every strategy x. (This game is also described in DS's Example 5, p. 2011.) Clearly, every strategy is a never-best reply and is dominated *only* by a dominated strategy. Eliminate

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[4] Formally, given any product subset $\widehat{S}$ of strategy profiles, Milgrom and Roberts (1990, p. 1265) defined the set of player i's *undominated responses* to $\widehat{S}$ as including strategies of i that are undominated by not only uneliminated strategies, but also by previously eliminated strategies. From the viewpoint of learning theory, the second feature of IESDS* can be "justified" by Milgrom and Roberts' (1990, p. 1269) adaptive learning process, where each player will never play a strategy for which there is another strategy, from the player's strategy space, that would have done better against every combination of the other players' strategies in the recent past plays.

[5] Chen and Luo (2003, Lemma 5) showed a similar result. Milgrom and Roberts (1996, Lemma 1, p. 117) proved an analogous result which allows for dominance by mixed strategy.

in round one all strategies except a particular strategy x in $(0, 1)$. Under DS's IESDS procedure, x survives DS's IESDS and is thus a "spurious Nash equilibrium." Under our IESDS*, in round two, x is further eliminated, and thus our maximal reduction yields an empty set of strategies, indicating (correctly) that the game has no Nash equilibrium. This makes sense since x cannot be justified as a best reply (and hence cannot be justified by any higher order knowledge of "rationality").[6] Consequently, this example shows that eliminating dominated strategies, rather than eliminating only those strategies that are dominated by some uneliminated strategy or by some undominated strategy, is a very natural and desirable requirement for a definition of IESDS in general games; see also our Example 2 in Section 2.

We also study the relationship between Nash equilibria and IESDS*. Example 4 in Section 3 demonstrates that, even with its strong elimination power, our IESDS* might generate spurious Nash equilibria. In particular, the game in Example 4 is in the class of Reny's (1999) better-reply secure games, which have regular properties such as compact and convex strategy spaces, as well as quasi-concave and bounded payoff functions. We do obtain

[6] The conventional notion of rationality requires that an individual's choice be optimal *within the feasible choice set* given his information; see Aumann (1987), Aumann and Brandenburger (1995), Bernheim (1984), Brandenburger and Dekel (1987), Epstein (1997), and Tan and Werlang (1988). In the case of finite games with continuous payoff functions, it is easy to see that n-level justifiable* strategy (meaning a player's choice is optimal *in the player's feasible strategy set* for some belief about the opponents' $(n-1)$-level justifiable* strategies) coincides with n-level justifiable strategy (meaning a player's choice is optimal *in the player's* $(n-1)$*-level justifiable strategy set* for some belief about the opponents' $(n-1)$-level justifiable strategies); see Pearce (1984, Proposition 2) and Osborne and Rubinstein (1994, Proposition 61.2). This coincidence makes it possible to define an alternative iteration for finite games *by gradually reduced subgames*. However, Osborne and Rubinstein (1994, Definitions 54.1 and 55.1) define rationalizability by the standard "best responses" *over the set of all feasible strategies.*

positive results: if the best replies are well defined, then no spurious Nash equilibria appear under IESDS*. In particular, no spurious Nash equilibria appear in one-person or "dominance solvable" games (see Theorem 4). Moreover, no spurious Nash equilibria appear in many games that arise in economic applications (see Corollary 4).

The remainder of this paper is organized as follows. Section 2 offers the definition of IESDS* and investigates its properties. Section 3 studies the relationship between IESDS* and Nash equilibria. Section 4 offers some concluding remarks. To facilitate reading, all the proofs are relegated to the Appendix.

2 IESDS*

Throughout this paper, we consider a *strategic game* $\mathcal{G} \equiv \left(N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where N is an arbitrary set of players, for each $i \in N$, X_i is an arbitrary set of player i's strategies, and $u_i : X_i \times X_{-i} \to \Re$ is i's arbitrary payoff function. $X \equiv \Pi_{i\in N} X_i$ is the joint strategy set. A strategy profile $x^* \in X$ is said to be a *Nash equilibrium* if for every i, x_i^* maximizes $u_i(., x_{-i}^*)$.

A strategy $x_i \in X_i$ is said to be *dominated given* $Y \subseteq X$ if for some strategy $x_i' \in X_i$,[7] $u_i(x_i', y_{-i}) > u_i(x_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$, where $Y_{-i} \equiv \{y_{-i}|$

[7] In the literature, especially in the case of finite games, a dominated (pure) strategy is normally defined by the existence of a mixed strategy that generates a higher expected payoff against any strategy profile of the opponents. In this paper, we follow DS in defining, rather conservatively, a dominated (pure) strategy by the existence of a (pure) strategy that generates a higher payoff against any strategy profile of the opponents. The two definitions of dominance are equivalent for games where strategy spaces are convex; for instance, mixed extensions of finite games. Borgers (1993) provided an interesting justification for "pure strategy dominance" by viewing players' payoff functions as preference orderings over the pure strategy outcomes of the game.

$(y_i, y_{-i}) \in Y\}$.

The following example illustrates that for some games, our IESDS* (a formal definition of which will be given below) yields a maximal reduction containing all Nash equilibria (in this case, a singleton) *only after an uncountably infinite number of rounds.* (This is unlike Lipman's (1994) Example and DS's Examples 3 and 6, which can be remedied to yield a maximal reduction by performing a second countable elimination after a first countable elimination.)

Example 1. Consider a two-person symmetric game: $\mathcal{G} \equiv \left(N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and for all $x_i, x_j \in [0, 1]$, $i, j = 1, 2$, and $i \neq j$

$$u_i(x_i, x_j) = \begin{cases} 1, & \text{if } x_i = 1 \\ 2, & \text{if } x_i \succ x_j \text{ and } x_i \neq 1 \\ 0, & \text{if } x_i \prec x_j \text{ or } x_i = x_j \neq 1 \end{cases},$$

where 4 is a linear order on $[0, 1]$ satisfying (i) 1 is the greatest element; and (ii) $[0, 1]$ is well ordered by the linear order 4.[8]

In this example only the least element r^0 in $[0, 1]$ (w.r.t. 4) is strictly dominated by 1. After eliminating r^0 from $[0, 1]$, only the least element r^1 in $[0, 1] \setminus \{r^0\}$ is strictly dominated by 1 given $[0, 1] \setminus \{r^0\}$. It is easy to see that every strategy is eliminated whenever every smaller strategy is eliminated and only one element in $[0, 1]$ is eliminated at each round. Thus, IESDS*

[8] A *linear order* is a *complete*, *reflexive*, *transitive*, and *antisymmetric* binary relation. A set is said to be *well ordered* by a linear order if each of its nonempty subsets has a least or first element. By the well-ordering principle — i.e., every nonempty set can be well ordered (see, e.g., Aliprantis and Border 1999, Section 1.12), $[0, 1)$ can be well ordered by a linear order . . The desired linear order 4 on $[0, 1]$ can be defined as: for any $r, r' \in [0, 1)$, $r \prec 1$; 1 4 1; and r 4 r' iff r . r'.

leads to a unique uncountable elimination, which leaves only the greatest element 1 for each player.[9]

Let us proceed to a formal definition of our IESDS*. For any subsets $Y, Y' \subseteq X$ where $Y' \subseteq Y$, we use the notation $Y \to Y'$ (read: Y is reduced to Y') to signify that for any $y \in Y \backslash Y'$, some y_i is dominated given Y. Let λ^0 denote the *first* element in an *ordinal* Λ, and let $\lambda+1$ denote the *successor* to λ in Λ.[10]

Definition. An *iterated elimination of strictly dominated strategies (IESDS*)* is defined as a finite, countably infinite, or uncountably infinite family $\left\{\mathcal{D}^{\lambda}\right\}_{\lambda\in\Lambda}$ such that $\mathcal{D}^{\lambda^0} = X$ (and $\mathcal{D}^{\lambda} = \cap_{\lambda'<\lambda}\mathcal{D}^{\lambda'}$ for a limit ordinal λ), $\mathcal{D}^{\lambda} \to \mathcal{D}^{\lambda+1}$, and $\mathcal{D} \equiv \cap_{\lambda\in\Lambda}\mathcal{D}^{\lambda} \to \mathcal{D}'$ only for $\mathcal{D}' = \mathcal{D}$. The set $\mathcal{D}$ is called a *"maximal reduction."*

The above definition of IESDS* does not require the elimination of *all* dominated strategies in each round of elimination. That is, we do not require that for every λ, $\mathcal{D}^{\lambda+1} = \mathcal{D}^{\lambda}\backslash\left\{y \in \mathcal{D}^{\lambda} \mid \exists i \text{ s.t. } y_i \text{ is dominated given } \mathcal{D}^{\lambda}\right\}$. This flexibility raises an important question: does the IESDS* procedure yield a unique maximal reduction? Without imposing any topological condition on the games, we show that IESDS* is always a well-defined order independent procedure and $\mathcal{D}$ is nonempty if a Nash equilibrium exists. Formally, we have:

[9] Example 1 also illustrates that DS's IESDS procedure may fail to yield a maximal reduction. DS's Theorem 1 on existence and uniqueness of maximal reduction relies on the game $\mathcal{G}$ being a compact and continuous game, which is not the case in our example (because it is impossible to find a topology on $[0,1]$ such that $\mathcal{G}$ is a compact and continuous game).

[10] An ordinal Λ is a well-ordered set in the order-isomorphic sense (see, e.g., Suppes 1972, p. 129). A *limit ordinal* is an element in Λ which is not a successor. As usual, we use $\lambda' < \lambda$ to means that "λ' precedes λ."

Theorem 1 *$\mathcal{D}$ uniquely exists. Moreover, $\mathcal{D}$ is nonempty if the game $\mathcal{G}$ has a Nash equilibrium.*

An immediate corollary of the proof of Theorem 1 is as follows:

Corollary 1. *Every Nash equilibrium survives both IESDS* and DS's IESDS procedures.*

In contrast to DS's IESDS, our IESDS* does not require that, in each round of elimination, the dominator of an eliminated strategy be some uneliminated strategy. However, the following result asserts that, for the class of games where strategy spaces are compact (Hausdorff) and payoff functions are uppersemicontinuous in own strategies, any maximal reduction of $\mathcal{G}$ using DS's IESDS procedure yields a joint strategy set identical to our $\mathcal{D}$. Thus, our IESDS* extends DS's IESDS to arbitrary games. Let $\mathcal{H}$ denote a maximal reduction of $\mathcal{G}$ in the DS sense, i.e., a set of strategy profiles resulting from using DS's IESDS procedure. Formally, we have:

Theorem 2 *For any compact and own-uppersemicontinuous game, $\mathcal{H} = \mathcal{D}$ if $\mathcal{H}$ exists. Moreover, for any compact (Hausdorff) and continuous game, $\mathcal{H} = \mathcal{D}$.*

The following example demonstrates that outside the class of compact and own-uppersemicontinuous games, $\mathcal{D}$ could be very different from a unique $\mathcal{H}$ that results from a well-defined "fast" IESDS procedure in the DS sense.

Example 2. Consider a two-person symmetric game: $\mathcal{G} = \left(N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1,2\}$, $X_1 = X_2 = [0,1]$, and for all $x_i, x_j \in [0,1]$, $i,j = 1,2$, and $i \neq j$ (cf. Fig. 1)

$$u_i(x_i,x_j) = \begin{cases} x_i, & \text{if } x_i < \frac{1}{2} \text{ or } x_j = \frac{1}{2} \\ \frac{1}{2}\min\{x_i,x_j\}, & \text{if } x_i \geq \frac{1}{2} \text{ and } x_j > \frac{1}{2} \\ 0, & \text{if } x_i \geq \frac{1}{2} \text{ and } x_j < \frac{1}{2} \end{cases}.$$

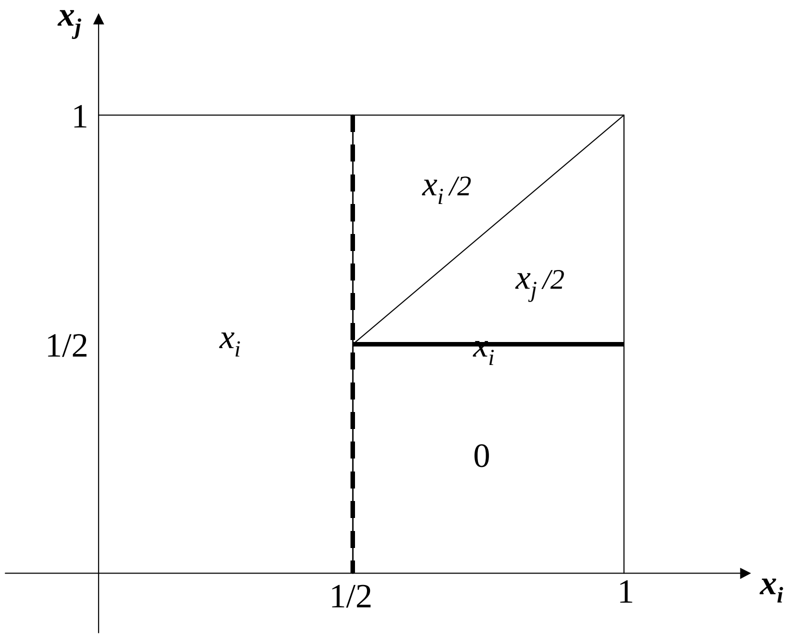

Fig. 1. Payoff function $u_i(x_i,x_j)$.

In this game it is easy to see that any strategy x_i in $[0,1/2)$ is dominated by $y_i = \frac{1}{4} + \frac{x_i}{2} > x_i$ for $x_i < \frac{1}{2}$. After eliminating all these dominated strategies, $1/2$ is dominated by 1 since (i) $u_i(1,1/2) = 1 > 1/2 = u_i(1/2,1/2)$ if $x_j = 1/2$, and (ii) $u_i(1,x_j) = x_j/2 > 1/4 = u_i(1/2,x_j)$ if $x_j > 1/2$. After eliminating the strategy $1/2$, no $x_i \in (1/2,1]$ is strictly dominated by some strategy $x_i' \in (1/2, 1]$, because in the joint strategy set $(1/2, 1] \times (1/2, 1]$, setting $x_j = x_i$, we have $u_i(x_i, x_j) = x_i \geq u_i(x_i', x_j)$ for all $x_i' \in (1/2, 1]$. Thus, $\mathcal{H} \equiv (1/2, 1] \times (1/2, 1]$ is a unique maximal reduction under the "fast" IESDS procedure in the DS sense.

However, any $x_i \in (1/2, 1)$ is dominated by the previously eliminated strategy $y_i = (1 + x_i)/4 \in [0, 1/2)$ since, for all $x_j \in (1/2, 1]$, $u_i(x_i, x_j) \leq x_i/2 < (1 + x_i)/4 = u_i(y_i, x_j)$. Thus, $\mathcal{D} = \{(1, 1)\} \neq \mathcal{H}$. In fact, $(1, 1)$ is the unique Nash Equilibrium, which could also be obtained with the "iterated elimination of never-best replies" (cf., e.g., Bernheim 1984; Lipman 1994). In this game, the payoff function $u_i(., x_j)$ is not uppersemicontinuous since $\limsup_{x_i \uparrow 1/2} u_i(x_i, x_j) = 1/2 > 0 = u_i(1/2, x_j)$ for all $x_j < 1/2$.

We end this section by providing a characterization of IESDS* by means of a "stability" criterion. A subset $\mathcal{K} \subseteq X$ is said to be a *stable set* if $\mathcal{K} = \{x \in X |\ x_i \text{ is not dominated given } \mathcal{K}\}$; cf. Luo (2001, Definition 3).

Theorem 3 *$\mathcal{D}$ is the largest stable set.*

The following result is an immediate corollary of Theorem 3.

Corollary 2. *Given a game $\mathcal{G}$, DS's IESDS is order independent if every $\mathcal{H}$ is a stable set.*

Corollary 2 does not require the game $\mathcal{G}$ to have compact strategy sets or uppersemicontinuous payoff functions. Of course, if strategy spaces are compact (Hausdorff) and payoff functions are uppersemicontinuous in own strategies, then by DS's Lemma, every dominated strategy has an undominated dominator. Under these conditions, every DS's maximal reduction $\mathcal{H}$ is a stable set. Corollary 2 therefore generalizes DS's Theorem 1(a). The following example illustrates this point.

Example 3. Consider a two-person game: $\mathcal{G} = \left(N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and for all $x_1, x_2 \in [0, 1]$, $u_1(x_1, x_2) = x_1$ and

$$u_2(x_1, x_2) = \begin{cases} x_2, & \text{if } x_2 < 1 \\ 1, & \text{if } x_1 = 1 \text{ and } x_2 = 1 \\ 0, & \text{if } x_1 < 1 \text{ and } x_2 = 1 \end{cases}.$$

In this example it is easy to see that $\{(1, 1)\}$ is the unique maximal reduction under DS's IESDS procedure, and that it is a stable set. Thus, DS's IESDS procedure is order independent in this game. However, $u_2(x_1, .)$ is not uppersemicontinuous in x_2 at $x_2 = 1$ if $x_1 \neq 1$ and hence DS's Theorem 1(a) does not apply.

Gilboa, Kalai, and Zemel (1990) (GKZ) considered a variety of elimination procedures. GKZ's definition of IESDS requires that in each round of elimination, any eliminated strategy is dominated by a strategy which is not eliminated in that round of elimination (see DS 2002, pp. 2018-2019). An immediate corollary of Theorems 1 and 3 is as follows.

Corollary 3. (i) *GKZ's IESDS procedure is order independent if every maximal reduction under GKZ's IESDS procedure is a stable set. (ii) In particular, for any compact and own-uppersemicontinuous game, both GKZ's IESDS procedure and our IESDS* procedure yield the same $\mathcal{D}$.*

3 IESDS* and Nash Equilibrium

As Nash (1950, p. 292) pointed out, "*no equilibrium point can involve a dominated strategy*". Nash equilibrium is clearly related to the notion of dominance. In this section we study the relationship between Nash equilibrium and IESDS*.

We have shown in Corollary 1 that every Nash equilibrium survives IESDS* and hence remains a Nash equilibrium in the reduced game after the iterated elimination procedure. The converse is not true in general. DS showed by examples (see their Examples 1, 4, 5, and 8) that their IESDS procedure can generate spurious Nash equilibria. Since our IESDS* has more elimination power, it can be easily verified that if we apply our IESDS* to DS's examples, there are no spurious Nash equilibria. Despite this happy outcome, the following example shows that IESDS* might generate spurious Nash equilibria.

Example 4. Consider a two-person symmetric game: $\mathcal{G} \equiv \left(N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and for all $x_i, x_j \in [0, 1]$, $i, j = 1, 2$, and $i \neq j$ (cf. Fig. 2)

$$u_i(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \in [1/2, 1] \text{ and } x_j \in [1/2, 1] \\ 1 + x_i, & \text{if } x_i \in [0, 1/2) \text{ and } x_j \in (2/3, 5/6) \\ x_i, & \text{otherwise} \end{cases} .$$

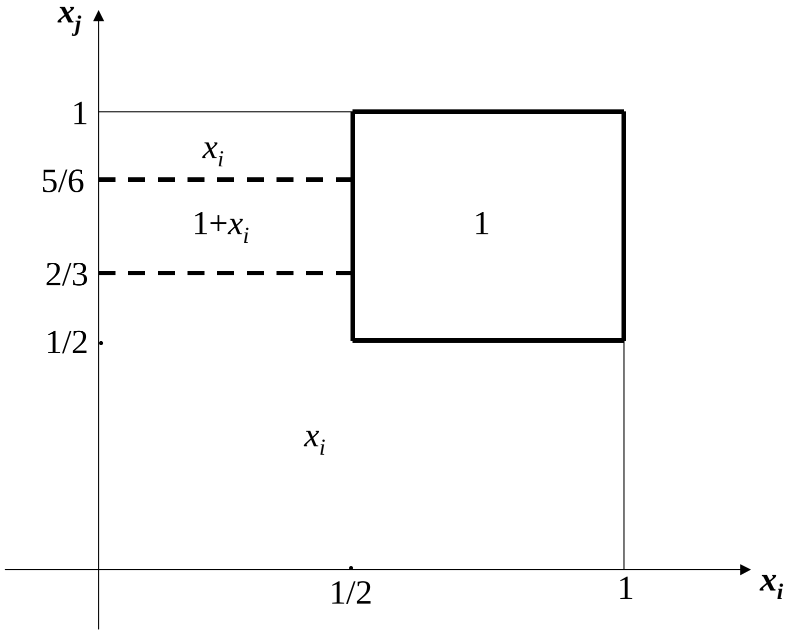

Fig. 2. Payoff function $u_i\left(x_i, x_j\right)$.

It is easily verified that $\mathcal{D} = [1/2, 1] \times [1/2, 1]$ since any $y_i \in [0, 1/2)$ is dominated. That is, IESDS* leaves the reduced game $\mathcal{G}|_{\mathcal{D}} \equiv \left(N, \{\mathcal{D}_i\}_{i\in N}, \{u_i|_{\mathcal{D}}\}_{i\in N}\right)$ that cannot be further reduced, where $u_i|_{\mathcal{D}}$ is the payoff function u_i restricted on $\mathcal{D}$. Clearly, $\mathcal{D}$ is the set of Nash equilibria in the reduced game $\mathcal{G}|_{\mathcal{D}}$ since $u_i|_{\mathcal{D}}$ is a constant function. However, it is easy to see that the set of Nash equilibria in game $\mathcal{G}$ is $\{x \in \mathcal{D}|\ x_1, x_2 \notin (2/3, 5/6)\}$. Thus, IESDS* generates spurious Nash equilibria $x \in \mathcal{D}$ where some $x_i\ \in (2/3, 5/6)$.

Remark. Example 4 belongs to Reny's (1999) class of games for which a Nash equilibrium exists (in this class of games, the player set is finite, the strategy sets are compact and convex, payoff functions are quasi-concave in own strategies, and a condition called "better-reply security" holds). To see that game $\mathcal{G}$ in Example 4 belongs to Reny's class of games, let us check the better-reply secure property. Recall that better-reply security means that "for every non equilibrium strategy x^* and every payoff vector limit u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^* (Reny 1999, p. 1030)". Let $\epsilon > 0$ be sufficiently small. We consider the following two cases:

(1) If $x^* \notin \mathcal{D}$, then some $x_i^* < 1/2$. Thus, i can secure payoff $x_i^* + \epsilon > u_i^* = x_i^*$ (if $x_j^* \notin (2/3, 5/6)$) or $x_i^* + 1 + \epsilon > u_i^* = x_i^* + 1$ (if $x_j^* \in (2/3, 5/6)$) by choosing a strategy $x_i^* + \epsilon$.

(2) If $x^* \in \mathcal{D}$, then some $x_i^* \in (2/3, 5/6)$ and $x_j^* \geq 1/2$. We distinguish two subcases: (2.1) $x_j^* > 1/2$. As x_i^* lies in an open interval $(2/3, 5/6)$, j can secure payoff $1 + x_j > 1$ by choosing a strategy $x_j \in (0, 1/2)$. (2.2) $x_j^* = 1/2$. In this subcase, the limiting vector u^* depends on how x approaches x^*. We must distinguish two subsubcases. (2.2.1) $u^* = (1, 1)$. Similarly to (2.1), j can secure payoff $1 + x_j > 1$ by choosing a strategy $x_j \in (0, 1/2)$. (2.2.2) The limiting payoff vector is $u^* = (x_i^*, 3/2)$ even though the actual payoff vector at $x^* \in \mathcal{D}$ is $(1, 1)$. Thus, i can secure payoff $x_i^* + \epsilon > u_i^* = x_i^*$ by choosing a strategy $x_i^* + \epsilon$, since for any x_j that deviates slightly from $1/2$,

$$u_i\left(x_i^* + \epsilon, x_j\right) = \begin{cases} x_i^* + \epsilon, & \text{if } x_j < 1/2 \\ 1, & \text{if } x_j \geq 1/2 \end{cases}.$$

Moreover, the player set is finite, strategy set $X_i = [0, 1]$ is compact and convex, and payoff function $u_i(\cdot, x_j)$ is quasi-concave and bounded. This example shows that IESDS* might generate spurious Nash equilibria in the class of Reny's better-reply secure games.

We next provide sufficient conditions under which IESDS* preserves the set of Nash equilibria. Consider a game $\mathcal{G} \equiv \left(N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$. We say that $\mathcal{G}$ has "well-defined best replies" if for every $i \in N$ and for every $x_{-i} \in X_{-i}$, there exists $x_i \in X_i$ that maximizes $u_i(., x_{-i})$. We say that $\mathcal{G}$ is "dominance-solvable" if IESDS* leads to a unique strategy choice for each player.[11] The following Theorem 4 asserts that (i) IESDS* cannot generate spurious Nash equilibria if the game has well-defined best replies, and that (ii) for one-person games and for dominance-solvable games, the set of Nash equilibria is identical to the set $\mathcal{D}$ generated by our IESDS*. Formally, let $\mathcal{NE}$ denote the set of Nash equilibria in $\mathcal{G}$, and let $\mathcal{NE}|_{\mathcal{D}}$ denote the set of Nash equilibria in the reduced game $\mathcal{G}|_{\mathcal{D}} \equiv \left(N, \{\mathcal{D}_i\}_{i\in N}, \{u_i|_{\mathcal{D}}\}_{i\in N}\right)$, where $u_i|_{\mathcal{D}}$ is the payoff function u_i restricted on $\mathcal{D}$. We can then state:

Theorem 4 *(i) If $\mathcal{G}$ has well-defined best replies, then $\mathcal{NE} = \mathcal{NE}|_{\mathcal{D}}$. (ii) If $\mathcal{G}$ is a one-person or dominance-solvable game, then $\mathcal{NE} = \mathcal{D}$.*

Remark. Theorem 4(i) is true also for DS's IESDS procedure (see DS 2002, Theorem 2). However, Theorem 4(ii) is not true for DS's IESDS procedure. To see this, consider again the one-person game in the Introduction. Clearly, no Nash equilibrium exists in the example. Because any strategy $x \in (0, 1)$ can survive DS's IESDS procedure, $\mathcal{H} = \{x\} \neq \mathcal{NE}$. This one-person game also demonstrates in the simplest manner that DS's IESDS procedure can generate spurious Nash equilibria.

Many important economic applications such as the Cournot game are

[11] For example, the standard Cournot game (Moulin, 1984), Bertrand oligopoly with differentiated products, the arms-race games (Milgrom and Roberts, 1990), and global games (Carlsson and van Damme, 1993).

dominance-solvable. As DS's Examples 3 and 8 illustrate, their IESDS procedure may fail to yield a maximal reduction and may produce spurious Nash equilibria in the Cournot game. By our Theorem 4, the set of Nash equilibria can be solved by our IESDS* in the class of dominance-solvable games. The following example, taken from DS's Example 3, illustrates this point.

Example 5 (Cournot competition with outside wager). Consider a three-person game $\mathcal{G} \equiv \left(N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1,2,3\}$, $X_1 = X_2 = [0,1]$, $X_3 = \{\alpha, \beta\}$, and for all x_1, x_2, and x_3, $u_1(x_1,x_2,x_3) = x_1(1-x_1-x_2)$, $u_2(x_1,x_2,x_3) = x_2(1-x_1-x_2)$, and

$$\begin{cases} u_3(x_1,x_2,\alpha) > u_3(x_1,x_2,\beta), & \text{if } (x_1,x_2) = (1/3,1/3) \\ u_3(x_1,x_2,\alpha) < u_3(x_1,x_2,\beta), & \text{otherwise} \end{cases}.$$

This game is dominance-solvable since our IESDS* yields $(1/3, 1/3, \alpha)$, which is the unique Nash equilibrium. By contrast, DS's IESDS procedure fails to give a maximal reduction since no countable sequence of elimination can eliminate the strategy β for player 3.

We close this section by listing the "preserving Nash equilibria" results for our IESDS* in some classes of games commonly discussed in the literature. These results follow immediately from Theorem 4(i).

Corollary 4. *$\mathcal{D}$ preserves the (nonempty) set of Nash equilibria in the following classes of games $\mathcal{G} \equiv \left(N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$:*

(i) **(Debreu 1952; Fan 1966; Glicksberg 1952).** *X_i is a nonempty, convex, and compact Hausdorff topological vector space; u_i is quasi-concave on X_i and continuous on $X_i \times X_{-i}$.*

(ii) **(Dasgupta and Maskin 1986).** *N is a finite set; X_i is a nonempty, convex, and compact space in a finite-dimensional Euclidian space; u_i is quasi-concave on X_i, uppersemicontinuous on $X_i \times X_{-i}$, and graph continuous.*

(iii) **(Topkis 1979; Vives 1990; Milgrom and Roberts 1990).** *$\mathcal{G}$ is a supermodular game such that X_i is a complete lattice; and u_i is order upper-semi-continuous on X_i and is bounded above.*

4 Concluding Remarks

We have presented a new notion of IESDS for general games, denoted by IESDS* and reflecting common knowledge of rationality. We show that IESDS* is always a well-defined order independent procedure, and that it can be used to identify Nash equilibrium in dominance-solvable games; e.g., the Cournot competition, Bertrand oligopoly with differentiated products, and the arms-race games. Many game theorists do not recommend iterated elimination of weakly dominated strategies (IEWDS) as a solution concept, and one important reason is that order matters for that procedure in some games (see, e.g., Marx and Swinkels 1997). This criticism cannot be applied to our IESDS*. As our IESDS* and DS's IESDS procedure lead to the same maximal reduction for the compact and continuous class of games, IESDS* can be viewed as an alternative to DS's IESDS procedure for games that are not compact and own-uppersemicontinuous. We have also characterized IESDS* as the largest stable set. This characterization suggests an interesting alternative definition of IESDS*.

While every Nash equilibrium survives IESDS*, we have demonstrated by Example 4 that IESDS* might generate spurious Nash equilibria. One remarkable feature of this example is that strategies eliminated by IESDS* are dominated by no strategy surviving IESDS* (and thus IESDS* creates spurious Nash equilibria in the reduced game). The creation of spurious Nash equilibria by IESDS seems to be an inherent and inevitable attribute of games that are not compact and own-uppersemicontinuous. The game in Example 4 belongs to Reny's (1999) class of games for the existence of a Nash equilibrium. Similarly to DS (2002, Theorem 2), we have shown that under IESDS*, the well-defined best replies property is sufficient to ensure the non-existence of spurious Nash equilibria. In particular, IESDS* never generates spurious Nash equilibria in dominance-solvable games. We have also pointed out that for many classes of games commonly discussed in the literature, IESDS* preserves Nash equilibria.

Finally, we would like to point out that in a related paper, Apt (2005) investigated the problem of order independence for "(possibly transfinite) iterated elimination of never-best replies." Apt (2005, Theorem 4.2) also showed a similar type of order-independent result and demonstrated that the result fails to hold for some different iteration procedures.

Appendix: Proofs

Theorem 1. *$\mathcal{D}$ uniquely exists. Moreover, $\mathcal{D}$ is nonempty if the game $\mathcal{G}$ has a Nash equilibrium.*

To prove Theorem 1, we need the following two lemmas:

Lemma 1 *For every $x \in \mathcal{D}$ and every i, x_i is not dominated given $\mathcal{D}$.*

Proof. Assume, in negation, that for some $y \in \mathcal{D}$ and some i, y_i is dominated given $\mathcal{D}$. Thus, $\mathcal{D} \to \mathcal{D} \backslash \{y\} \neq \mathcal{D}$, which is a contradiction. ■

Lemma 2 *For any $Y \subseteq Y'$, a strategy is dominated given Y if it is dominated given Y'.*

Proof. Let y_i be a strategy that is dominated given Y'. That is, $u_i(x_i, y_{-i}) > u_i(y_i, y_{-i})$ for some $x_i \in X_i$ and all $y_{-i} \in Y'_{-i}$. Since $Y \subseteq Y'$, $u_i(x_i, y_{-i}) > u_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$. Therefore, y_i is dominated given Y. ■

We now turn to the proof of Theorem 1.

Proof of Theorem 1. For any $Y \subseteq X$ define the "next elimination" operation ∇ by

$$\nabla [Y] \equiv \{y \in Y | \ \exists i \text{ s.t. } y_i \text{ is dominated given } Y\}.$$

The existence of a maximal reduction using IESDS* is assured by the following prominent "fast" IESDS*: $\mathcal{D} \equiv \cap_{\lambda \in \Lambda} \mathcal{D}^{\lambda}$ satisfying $\mathcal{D}^{\lambda+1} = \mathcal{D}^{\lambda} \backslash \nabla \left[\mathcal{D}^{\lambda}\right]$, where Λ is an ordinal that is order-isomorphic, via an isomorphism φ, to the well-ordered quotient X/∇ (more specifically, $\varphi(\lambda+1) = \nabla \left[\mathcal{D}^{\lambda}\right]$ with $\varphi\left(\lambda^0\right) = \emptyset$, and the linear order on the quotient X/∇ can be defined by the obvious "next elimination" relation $\nabla [\cdot]$). Now suppose that $\mathcal{D}$ and $\mathcal{D}'$ are two maximal reductions obtained by applying IESDS* procedure. Since $\mathcal{D} \cup \mathcal{D}' \subseteq \mathcal{D}^{\lambda^0}$, by Lemmas 1 and 2, $\mathcal{D} \cup \mathcal{D}' \subseteq \mathcal{D}^{\lambda}$ for *all* λ. Therefore, $\mathcal{D} \cup \mathcal{D}' \subseteq \mathcal{D}$. Similarly, $\mathcal{D} \cup \mathcal{D}' \subseteq \mathcal{D}'$. Thus, $\mathcal{D} = \mathcal{D}'$. Let x^* be a Nash equilibrium. Since for every i, x_i^* is not dominated given $\{x^*\}$, by Lemma 2, $x^* \in \mathcal{D}^{\lambda}$ for all λ. ■

Corollary 1. *Every Nash equilibrium survives both IESDS* and DS's IESDS procedures.*

Proof. Let $\mathcal{H}$ be the maximal reduction resulting from an IESDS procedure in the DS sense. Since every strategy that is dominated by an uneliminated strategy is a dominated strategy, by Theorem 1, the unique $\mathcal{D} \subseteq \mathcal{H}$. By the proof of Theorem 1, every Nash equilibrium survives $\mathcal{D}$ and hence, survives $\mathcal{H}$. ■

Theorem 2. *For any compact and own-uppersemicontinuous game, $\mathcal{H} = \mathcal{D}$ if $\mathcal{H}$ exists. Moreover, for any compact (Hausdorff) and continuous game, $\mathcal{H} = \mathcal{D}$.*

Proof. Suppose that $\mathcal{H}$ is the maximal reduction resulting from an IESDS procedure in the DS sense. Clearly, $\mathcal{D} = \emptyset$ if $\mathcal{H} = \emptyset$. By DS's Lemma, for any i and any $x \in \mathcal{H} \neq \emptyset$, x_i is not dominated given $\mathcal{H}$. According to Definition in this paper, $\mathcal{H} = \mathcal{D}$. Moreover, by DS's Theorem 1(b), $\mathcal{H}$ exists if the game is compact and continuous. The last part of Theorem 2 follows immediately from the first part and from DS's Theorem 1. ■

Theorem 3. *$\mathcal{D}$ is the largest stable set.*

Proof. By Lemma 1, $\mathcal{D}$ is a stable set. We must show that all stable sets are subsets of $\mathcal{D}$. By Lemma 2, every stable set $\mathcal{K} \subseteq \mathcal{D}^{\lambda}$ for *all* λ. Therefore, $\mathcal{D} \equiv \cap_{\lambda \in \Lambda} \mathcal{D}^{\lambda}$ is the largest stable set. ■

Corollary 2. *Given a game $\mathcal{G}$, DS's IESDS is order independent if every $\mathcal{H}$ is a stable set.*

Proof. Let $\mathcal{H}$ be a maximal reduction resulting from an IESDS procedure in the DS sense. It suffices to show that $\mathcal{H} = \mathcal{D}$. Since $\mathcal{H}$ is a stable set, by Theorem 3, $\mathcal{H} \subseteq \mathcal{D}$. By the proof of Corollary 1, $\mathcal{D} \subseteq \mathcal{H}$. Therefore, $\mathcal{H} = \mathcal{D}$. ■

Corollary 3. (i) *GKZ's IESDS procedure is order independent if every maximal reduction under GKZ's IESDS procedure is a stable set. (ii) In particular, for any compact and own-uppersemicontinuous game, both GKZ's IESDS procedure and our IESDS* procedure yield the same $\mathcal{D}$.*

Proof. The proof of the first part is totally similar to the proof of Corollary 2. Now suppose that the game is compact and own-uppersemicontinuous, and suppose that $\mathcal{H}$ is a maximal reduction resulting from an IESDS procedure in the GKZ sense. Clearly, $\mathcal{D} = \emptyset$ if $\mathcal{H} = \emptyset$. By DS's Lemma and DS's Theorem 3, GKZ's definition of IESDS coincides with DS's definition of IESDS. By Theorem 2, $\mathcal{H} = \mathcal{D}$. ■

Theorem 4. *(i) If $\mathcal{G}$ has well-defined best replies, then $\mathcal{NE} = \mathcal{NE}|_{\mathcal{D}}$. (ii) If $\mathcal{G}$ is a one-person or dominance-solvable game, then $\mathcal{NE} = \mathcal{D}$.*

Proof. (i) Let $x^* \in \mathcal{NE}|_{\mathcal{D}}$. Since $\mathcal{G}$ has well-defined best replies, $u_i\left(x_i^{**}, x_{-i}^*\right) \geq u_i\left(x_i, x_{-i}^*\right)$ for some $x_i^{**} \in X_i$ and all $x_i \in X_i$. Therefore, for every i, x_i^{**} is not dominated given $\mathcal{D}$. Since $x^* \in \mathcal{D}$, by Theorem 3, $\left(x_i^{**}, x_{-i}^*\right) \in \mathcal{D}$. Since $x^* \in \mathcal{NE}|_{\mathcal{D}}$, it follows that $u_i\left(x_i^*, x_{-i}^*\right) \geq u_i\left(x_i^{**}, x_{-i}^*\right) \geq u_i\left(x_i, x_{-i}^*\right)$ for all $x_i \in X_i$. That is, $x^* \in \mathcal{NE}$. Thus, $\mathcal{NE}|_{\mathcal{D}} \subseteq \mathcal{NE}$. By Corollary 1, $\mathcal{NE} \subseteq \mathcal{NE}|_{\mathcal{D}}$. Hence, $\mathcal{NE}|_{\mathcal{D}} = \mathcal{NE}$.

(ii) By Corollary 1, $\mathcal{NE} \subseteq \mathcal{D}$. Therefore, $\mathcal{NE} = \mathcal{D}$ if $\mathcal{D} = \emptyset$. Now suppose $\mathcal{D} \neq \emptyset$. By Theorem 3, $\mathcal{D}$ is a stable set. Thus, for every $i \in N$ and for every $x^* \in \mathcal{D}$, $u_i\left(x^*\right) \geq u_i\left(x_i, x_{-i}^*\right)$ for all $x \in X_i$ if $\mathcal{G}$ is a one-person or dominance-solvable game. That is, every $x^* \in \mathcal{D}$ is a Nash equilibrium. Hence, $\mathcal{NE} = \mathcal{D}$. ■

Corollary 4. *$\mathcal{D}$ preserves the (nonempty) set of Nash equilibria in the following classes of games $\mathcal{G} \equiv \left(N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$:*

(i) **(Debreu 1952; Fan 1966; Glicksberg 1952).** *X_i is a nonempty, convex, and compact Hausdorff topological vector space; u_i is quasi-concave on X_i and continuous on $X_i \times X_{-i}$.*

(ii) **(Dasgupta and Maskin 1986).** *N is a finite set; X_i is a nonempty, convex, and compact space in a finite-dimensional Euclidian space; u_i is quasi-concave on X_i, uppersemicontinuous on $X_i \times X_{-i}$, and graph continuous.*

(iii) **(Topkis 1979; Vives 1990; Milgrom and Roberts 1990).** *$\mathcal{G}$ is a supermodular game such that X_i is a complete lattice; and u_i is order upper-semi-continuous on X_i and is bounded above.*

Proof. By the Generalized Weierstrass Theorem (see, e.g., Aliprantis and Border 1999, 2.40 Theorem), the best replies are well-defined for the compact and own-uppersemicontinuous games. By Milgrom and Roberts' (1990) Theorem 1, the best replies are well-defined for the supermodular games in which strategy spaces are complete lattices. By Theorem 4(i), IESDS* preserves the (nonempty) set of Nash equilibria for these classes of games in Corollary 4. ■

Acknowledgements: This paper partially done while the third author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2005. We would like to thank Krzysztof R. Apt, Yossi Greenberg, Tai-Wei Hu, Chenying Huang, Wei-Torng Juang, Huiwen Koo, Fan-Chin Kung, Kim Long, Man-Chung Ng, Yeneng Sun, Licun Xue, and Chun-Hsien Yeh for helpful discussions and comments. We especially thank Martin Dufwenberg for his encouragement and helpful comments. We thank participants in seminars at Academia Sinica, National University of Singapore, and the 16th International Conference on Game Theory, Stony Brook, NY. Financial supports from the National Science Council of Taiwan, and from SSHRC of Canada are gratefully acknowledged. The usual disclaimer applies.

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