Abstract

The low for random reals are characterized topologically, as well as in terms of domination of Turing functionals on a set of positive measure.

Low for random reals and positive-measure domination

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January 21, 2006

1 Introduction

A function $f : \omega \to \omega$ is *uniformly almost everywhere (a.e.) dominating* if for measure-one many X, and all g computable from X, f dominates g. Such functions were first studied by Kurtz [8] who showed that uniformly a.e. dominating functions exist and that in fact $0'$, the Turing degree of the halting problem, computes one of them. If we replace measure by category, there are no such functions, as is not hard to see. A few decades later Dobrinen and Simpson [5] made use of a.e. domination in Reverse Mathematics. They made a couple of fundamental conjectures that were promptly refuted in [2] and [4]. In this article we strengthen the results of [2] to provide a characterization of a related concept, positive-measure domination, in terms of lowness for randomness. Conversely, we characterize low for random reals in terms of such domination. The following characterizations are already known. (We assume the reader is familiar with the definition of Martin-Löf random reals and of prefix-free Kolmogorov complexity K.)

Theorem 1.1 (Nies, Hirschfeldt, Stephan, Terwijn [6],[10],[11]). *The following are equivalent for $A \in 2^\omega$:*

- *A is low for random: each Martin-Löf random real is Martin-Löf random relative to A.*
- *A is K-trivial:* $\exists c \forall n\, K(A \upharpoonright n) \le K(\emptyset \upharpoonright n) + c$.
- *A is low for K:* $\exists c \forall n\, K(n) \le K^A(n) + c$.
- $\exists Z \ge_T A$, *Z is ML-random relative to A.*
- $A \le_T 0'$ *and Ω is ML-random relative to A*

*The author thanks the Institute for Mathematical Sciences of the National University of Singapore for support during preparation of this manuscript at the *Computational Prospects of Infinity* conference in Summer 2005. The author also thanks Denis R. Hirschfeldt for proving upon request a lemma used in an earlier proof of the case $B \le_T 0'$ of Theorem 3.2.

The low for random reals induce a Σ_3^0 nonprincipal ideal in the Turing degrees bounded above by a low$_2$ Δ_2^0 degree [10], and have already found application to long-standing open problems in computability theory. Our characterizations in this paper are distinguished by not being couched in the language of randomness and Kolmogorov complexity. They do however refer to measure; it remains open whether a characterization purely in terms of domination or traces can be given such as that found for low for Schnorr random reals [1][13].

The first main result of Section 2 is Theorem 2.12, which is a characterization of the low for random reals in terms of containment of effectively closed sets of positive measure. Building on this result, Theorem 2.14 is a characterization of low for random reals in terms of positive-measure domination. Section 3 contains, first, a characterization of the Turing degrees relative to which $0'$ is low for random, in terms of positive-measure domination. Finally, with a view toward future research, we include a proof that there is a Turing functional that is universal for this kind of domination.

2 Low for random reals

To obtain our topological characterization, we will pass first from a certain universal Martin-Löf test (given in terms of K) to an arbitrary Martin-Löf test, and then to an arbitrary open set of measure < 1.

Theorem 2.1 (Kraft-Chaitin Theorem [3])**.** *Suppose $\langle n_k, \sigma_k \rangle$, $k \in \omega$ is a recursive sequence, with $\sum_k 2^{-n_k} \leq 1$. Then there exists a prefix-free machine M and a collection of strings τ_k with $|\tau_k| = n_k$ and $M(\tau_k) = \sigma_k$.*

Definition 2.2 (Chaitin)**.** Let $A \in 2^\omega$. An *information content measure relative to A* is a partial function $\hat{K} : 2^{<\omega} \to \omega$ such that

$$\sum_{\sigma \in 2^{<\omega}} 2^{-\hat{K}(\sigma)} \leq 1$$

and $\{\langle \sigma, k \rangle : \hat{K}(\sigma) \leq k\}$ is r.e. in A.

Lemma 2.3. *Let $\{V_n^A\}_{n \in \omega}$ be a Martin-Löf test for A-randomness. Then there is an information content measure $\hat{K}$ relative to A, such that $V_n^A = \{X : \exists m\ \hat{K}(X \upharpoonright m) \leq m - n\}$ for all n.*

Proof. Let σ_k be the kth string to enter V_n^A, and let $n_k = |\sigma_k| - n$. Note that $\sum_k 2^{-n_k} = 2^n \sum_k 2^{-|\sigma_k|} \leq 1$, so we can let M be as in Theorem 2.1. In other words, if σ enters V_n^A then produce an M-description of σ of length $\leq |\sigma| - n$. Let $\hat{K}(\sigma)$ be the length of the shortest description of σ so produced. □

Lemma 2.4 (Chaitin)**.** *If $\hat{K}$ is an information content measure relative to a real A, then for all n, $K^A(n) \leq \hat{K}(n) + \mathcal{O}(1)$.*

Definition 2.5. For any real X, let $S^A = \{S_n^A\}_{n \in \omega}$ where $S_n^A = \{X : \exists m\ K^A(X \upharpoonright m) \leq m - n\}$.

Lemma 2.6. *If V^A is a Martin-Löf test relative to A, then for each n there exists p such that $V^A_p \subseteq S^A_n$.*

Proof. Let $\hat{K}$ be as in Lemma 2.3. By Lemma 2.4, there is a constant d such that $K^A(\sigma) \leq \hat{K}(\sigma) + d$ for all σ. Hence given n, we let $m = n + d$. Then $\exists m\, \hat{K}(X \upharpoonright m) \leq m - p$ implies $\exists m\, K(X \upharpoonright m) \leq m - n$. $\square$

Corollary 2.7 (Schnorr [12])**.** *S^A is a universal Martin-Löf test relative to A.*

Proof. In Lemma 2.6 we can conclude that $\bigcap_n V^A_n \subseteq \bigcap_n S^A_n$. $\square$

Definition 2.8. Let $n \geq 1$. Let Σ^μ_n denote the collection of all Σ^0_n classes of measure < 1. The complement of a Σ^μ_n class is a Π^μ_n class. The complement of U is denoted $\overline{U}$. The clopen subset of 2^ω generated by $\sigma \in 2^{<\omega}$ is denoted $[\sigma]$, and concatenation of strings is denoted by juxtaposition.

If U, V are open subsets of 2^ω then their product is $UV = \bigcup\{[\sigma\tau] : [\sigma] \subseteq U,\ [\tau] \subseteq V\}$. We define $U^n = U^{n-1}U$ where $U^1 = U$. We can also think of this exponentiation as acting on a closed set Q, defining Q^n via the equation $\overline{Q^n} = \overline{Q}^n$. It will be clear whether we are considering a set as open or closed.

Lemma 2.9 (Kučera [7])**.** *For each $\Pi^\mu_1(A)$ class Q there is a computable function f such that $\{\overline{Q^{f(n)}}\}_{n\in\omega}$, is a Martin-Löf test relative to A.*

Proof. Let $q > 0$ be a rational number such that $\mu Q \geq q$. Let $P = \overline{Q}$. Then $\mu P^n = (\mu P)^n \leq (1 - q)^n$. Let f be a computable function such that for all $k \in \omega$, $\mu P^{f(k)} \leq 2^{-k}$. Let $V^A_k = P^{f(k)}$. Then V^A is a Martin-Löf test relative to A. $\square$

Lemma 2.10. *If P is an open set such that P^n is contained in a Σ^μ_1 class for some $n \geq 2$, then P itself is contained in a Σ^μ_1 class.*

Proof. We write $U|\sigma = \bigcup\{[\tau] : [\sigma\tau] \subseteq U\}$. Note that if P is open then so is P^2. Hence by iteration, it suffices to consider the case $n = 2$. So suppose $(\exists U)\ P^2 \subseteq U \in \Sigma^\mu_1$. Case 1: $\exists\sigma$, $\mu(U|\sigma) < 1$, $[\sigma] \subseteq P$. Then $P^2 \cap [\sigma] = [\sigma]P$, the product of $[\sigma]$ and P. Then $P = ([\sigma]P)|\sigma = (P^2 \cap [\sigma])|\sigma = P^2|\sigma \subseteq U|\sigma \in \Sigma^\mu_1$. Case 2: Otherwise; so $P \subseteq \bigcup\{[\sigma] : \mu(U|\sigma) = 1\}$. Fix $\epsilon > 0$ such that $\mu U < 1-\epsilon$, and let $V = \bigcup\{[\sigma] : \mu(U|\sigma) \geq 1 - \epsilon\}$. Note that V is Σ^0_1, contains P, and $\mu V < 1$ because $(1 - \epsilon)\mu V \leq \mu U < 1 - \epsilon$. $\square$

As usual, an A-random is a real that is Martin-Löf random relative to A. If $A, B \in 2^\omega$ then we A is a *tail* of B if there exists n such that $A(k) = B(n + k)$ for all $k \in \omega$.

Lemma 2.11 (Kučera [7])**.** *For each $A \in 2^\omega$, each A-random is a tail of an element of each $\Pi^\mu_1(A)$ class.*

Proof. Let Q be a $\Pi^\mu_1(A)$ class and suppose X is A-random. Then by Lemma 2.9, there is an m such that $X \in Q^m$. If $m = 2$ then clearly, as Q is closed, X is a tail of an element of Q. If $m > 2$, the result follows by iteration since each Q^m is closed. $\square$

Theorem 2.12. *Let $A \in 2^\omega$. The following are equivalent:*

1. *Each 1-random real is A-random (A is* low for random *[10]).*
2. *For each Π^μ_1 class Q consisting entirely of* 1*-random reals, there exist σ, n such that $Q \cap [\sigma] \neq \emptyset$ but $Q \cap S^A_n \cap [\sigma] = \emptyset$.*
3. *For some n, $\overline{S^A_n}$ has a Π^μ_1 subclass.*
4. *For each A-Martin-Löf test V^A_n, there exists an n such that $\overline{V^A_n}$ has a Π^μ_1 subclass.*
5. *For each $\Pi^\mu_1(A)$ class Q there exists an n such that Q^n has a Π^μ_1 subclass.*
6. *Each $\Pi^\mu_1(A)$ class has a Π^μ_1 subclass.*
7. *Some $\Pi^\mu_1(A)$ class consisting entirely of A-random reals has a Π^μ_1 subclass.*
8. *The class of A-random reals has a Π^μ_1 subclass.*

Proof. (1)⇒(2): For this implication we use an argument of Nies and Stephan [9]. Suppose A is low for random but (2) fails. So there is a Π^μ_1 class Q consisting entirely of 1-random reals, such that for all σ, n, if $Q \cap [\sigma] \neq \emptyset$ then $Q \cap S^A_n \cap [\sigma] \neq \emptyset$. Let $\sigma_0 = \lambda$, and $\sigma_{n+1} \succeq \sigma_n$, with $[\sigma_{n+1}] \subseteq S^A_n$ but $[\sigma_{n+1}] \cap Q \neq \emptyset$. Then $Y = \bigcup_{n\in\omega} \sigma_n$ is not A-random, but is 1-random, since $Y \in Q$. (2)⇒(3) Let Q be as in (2), and let n, σ be as guaranteed by (2) for Q. Then $Q \cap [\sigma]$ is the desired subclass. (3)⇒(4): Lemma 2.6. (4)⇒(5): Let Q be a $\Pi^\mu_1(A)$ class. By Lemma 2.9, $V^A_k = \overline{Q^{f(k)}}$ is a Martin-Löf test relative to A for some computable f. By (4), $Q^{f(m)} = \overline{V^A_m} \supseteq F$ for some $F \in \Pi^\mu_1$ and m; let $n = f(m)$. (5)⇒(6): Lemma 2.10. (6)⇒(7): If U^A is a universal Martin-Löf test for A-randomness then we can let $Q = \overline{U_1}$. (7)⇒(8): Since any class consisting entirely of A-randoms is contained in the class of all A-randoms. (8)⇒(1): Suppose X is 1-random; we need to show X is A-random. Let F be a Π^μ_1 subclass of the class of A-randoms. By Lemma 2.11, X is tail of an element of F. Hence X is a tail of an A-random, so X is A-random. □

To characterize the low for random reals in terms of domination we first introduce some notation. We write Tot(Φ)=$\{X : \Phi^X$ is total$\}$ and $\varphi^X(n) = (\mu s)(\forall m < n)(\Phi^X_s(m)\downarrow \leq s)$. Note that Tot($\Phi$) is a Π^0_2 class for each Φ, and Tot(Φ)=Tot(φ). The function φ is the running time of Φ, explicitly satisfying $\Phi^X(n) \leq \varphi^X(n)$ for all n. Let Φ be a Turing functional and $B \in 2^\omega$. If there exists $f \leq_T B$ such that for positive-measure many X, Φ^X is dominated (equivalently, majorized) by f, then we write $\Phi < B$.

Lemma 2.13 (implicit in [5])**.** *Let $B \in 2^\omega$ and let Φ be a Turing functional. Then $\varphi < B$ iff Tot(Φ) has a $\Pi^\mu_1(B)$ subclass.*

Proof. First suppose $\varphi < B$, as witnessed by f. Then $\{X : \forall n\ \Phi^X_{f(n)}(n)\downarrow\}$ is a $\Pi^\mu_1(B)$ subclass of Tot(Φ). Conversely, let F be a $\Pi^\mu_1(B)$ subclass of Tot(Φ). By compactness, $\{\varphi^X(n) : X \in F\}$ is finite for each n, and $\{\langle n, m\rangle : \forall X(X \in F \to \varphi^X(n) < m\}$ is a $\Sigma^0_1(B)$ class. Hence by $\Sigma^0_1(B)$ uniformization there is a function $f \leq_T B$ such that $\forall n \forall X(X \in F \to \varphi^X(n) < f(n))$; i.e., f witnesses that $\varphi < B$. ☐

Theorem 2.14. *Let $A \in 2^\omega$. The following are equivalent:*

1. *A is low for random.*
2. *Each $\Pi^\mu_1(A)$ class has a Π^μ_1 subclass.*
3. *(i) $A \leq_T 0'$ and (ii) for each Φ, if Tot(Φ) has a $\Pi^\mu_1(A)$ subclass then $\varphi < 0$.*
4. *(i) $A \leq_T 0'$, and (ii) for each Φ, if $\varphi < A$ then $\varphi < 0$.*

Proof. (1)⇔(2) was shown in Theorem 2.12. (2)⇒(3): Nies [10] shows that if A is low for random then $A \leq_T 0'$. Suppose Tot(Φ) has a $\Pi^\mu_1(A)$ subclass Q. By (2), Tot(Φ) has a Π^μ_1 subclass F. By Lemma 2.13, we are done. (3)⇒(2): Suppose (3) holds and suppose Q is a $\Pi^\mu_1(A)$ class. Pick Ψ such that $Q = \{X : \Psi^{X\oplus A}(0)\uparrow\}$. Since $A \leq_T 0'$, $A = \lim_s A_s$, the limit of a computable approximation. Let $\Phi^X(s) = \mu t > s(\Psi^{X\oplus A_t}_t(0)\uparrow)$. Then $Q =$Tot(Φ). Applying (3) to this Φ, we have $\varphi < 0$ and so by Lemma 2.13 we are done. (3)⇔(4) is immediate from Lemma 2.13. ☐

3 Positive-measure domination

In [5] it was asked whether the Turing degrees A of uniformly a.e. dominating functions are characterized by either of the inequalities $A \geq 0'$ and $A' \geq_T 0''$. The case $A \geq 0'$ was refuted by a direct construction in [4]. The case $A' \geq_T 0''$ was refuted in [2] using precursors to the results presented here. Namely, the dual of property 4(ii) above is $\forall\varphi(\varphi < A)$ or equivalently $\forall\varphi(\varphi < 0' \to \varphi < A)$. Relativizing our proofs gives that this is equivalent to: $0'$ is low for random relative to A. If we restrict ourselves to $A \leq_T 0'$, then by [10] this implies $A' \geq_{tt} 0''$, which is strictly stronger than $A' \geq_T 0''$. We do not know whether the assumption $A \leq_T 0'$ is necessary for either of the conclusions $A' \geq_{tt} 0''$, $A' \geq_T 0''$.

We say that A is *positive-measure dominating* if $\forall\varphi(\varphi < A)$. If each B-random real is A-random then we write $A \leq_{LR} B$ (A is low for random relative to B) following [10]. We write Φ^A for the functional $X \mapsto \Phi^{A\oplus X}$.

Lemma 3.1. *Let $A \in 2^\omega$ and $\mathcal{C} \subseteq 2^\omega$. Then $\mathcal{C}$ is a $\Pi^0_2(A)$ class iff $\mathcal{C}$ is Tot(Φ^A) for some Turing functional Φ.*

Proof. Suppose $\mathcal{C}$ is a $\Pi^0_2(A)$ class, i.e. $\mathcal{C} = \{X : \forall y \exists s R(y, s, A, X)\}$ where R is a formula in the language of second-order arithmetic all of whose quantifiers are first-order and bounded. Then we can let $\Phi^{A\oplus X}(y) = \mu s(R(y, s, A, X))$. Conversely, $\mathrm{Tot}(\Phi^A) = \{X : \forall y \exists s(\Phi^{A\oplus X}_s(y)\downarrow)\}$. □

Theorem 3.2. *Let $B \in 2^\omega$. Then $0'$ is low for random relative to B iff B is positive-measure dominating.*

Proof. This is the special case $A = 0$ of the fact that for each $A, B \in 2^\omega$, the following are equivalent:

1. $A' \leq_{LR} A \oplus B$.
2. Each $\Pi^\mu_1(A')$ class has a $\Pi^\mu_1(A \oplus B)$ subclass.
3. Each $\Pi^\mu_2(A)$ class has a $\Pi^\mu_1(A \oplus B)$ subclass.
4. $\forall \Phi$, if $\mathrm{Tot}(\Phi^A)$ has positive measure then it has a $\Pi^\mu_1(A \oplus B)$ subclass.
5. $\forall \Phi(\varphi^A < A \oplus B)$

The equivalences are proved as follows. (1)⇔(2): Relativization of Theorem 2.12 gives: $A \leq_{LR} B$ iff each $\Pi^\mu_1(A)$ class has a $\Pi^\mu_1(B)$ subclass. (3)⇔(4): Lemma 3.1. (4)⇔(5): Relativization of Lemma 2.13. (2)⇔(3): Let $A \in 2^\omega$. A' is uniformly a.e. dominating relative to A, hence A' is positive-measure dominating relative to A. Hence by putting $B = A'$ in (3)⇔(5), each $\Pi^\mu_2(A)$ class has a $\Pi^\mu_1(A')$ subclass.

□

Universal functionals

Suppose Φ_i, $i \in \omega$ are all the Turing functionals. As observed in [4], the functional Ψ given by $\Psi^{0^i 1 X} = \Phi^X_i$ is *universal* for uniform a.e. domination, in the sense that any function that dominates Ψ on almost every X, is a uniformly a.e. dominating function. As $\Psi < 0$, Ψ is not universal for positive-measure domination; however, the following functional is.

Fix a constant $c \in \omega$. Let $\Xi^X_c(s)$ be the least stage $t > s$ at which X looks like it is 2-random, with constant c. That is,

$$\Xi^X_c(s) = (\mu t > s)(\forall n \leq t\ K^{0'_t}_t(X \upharpoonright n) \geq n - c).$$

Here $0'_t$ is the approximation to $0'$ at stage t, and K^A_t the approximation to K^A for any $A \in 2^\omega$. Considering S^X_c (Definition 2.5), it is clear that Ξ_c is total for positive-measure many X, all of which are 2-randoms. The running time ξ_c of Ξ_c is universal for positive-measure domination in the following sense.

Lemma 3.3. *For each $A \in 2^\omega$ and $c \in \omega$, if $\xi_c < A$ then A is positive-measure dominating.*

Proof. The complement of $\mathrm{Tot}(\Xi_c)$ is $\{X : \exists n\, K^{0'}(X \upharpoonright n) < n - c\}$ which is open. Hence $\mathrm{Tot}(\Xi_c)$ is closed and is in fact a $\Pi^\mu_1(0')$ class. Suppose $\xi_c < A$. By Lemma 2.13, $\mathrm{Tot}(\Xi_c)$ has a $\Pi^\mu_1(A)$ subclass. Thus: Some $\Pi^\mu_1(0')$ class consisting entirely of $0'$-randoms has a $\Pi^\mu_1(A)$ subclass. By Theorem 2.12 (7) relativized, $0' \leq_{LR} A$, and so by Theorem 3.2, A is positive-measure dominating. $\square$

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