Approximation of the solution and its derivative for the singularly perturbed Black-Scholes equation with nonsmooth initial data[*]

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A problem for the Black-Scholes equation that arises in financial mathematics, by a transformation of variables, is leaded to the Cauchy problem for a singularly perturbed parabolic equation with variables x, t and a perturbation parameter ε, $\varepsilon \in (0, 1]$. This problem has several singularities such as: the unbounded domain; the piecewise smooth initial function (its first order derivative in x has a discontinuity of the first kind at the point $x = 0$); an interior (moving in time) layer generated by the piecewise smooth initial function for small values of the parameter ε; etc.

In this paper, a grid approximation of the solution and its first order derivative is studied in a finite domain including the interior layer. On a uniform mesh, using the method of additive splitting of a singularity of the interior layer type, a special difference scheme is constructed that allows us to approximate ε-uniformly both the solution of the boundary value problem and its first order derivative in x with convergence orders close to 1 and 0.5, respectively. The efficiency of the constructed scheme is illustrated by numerical experiments.

1 Introduction

1.1. Mathematical modeling in financial mathematics leads to the Black-Scholes equation (that is backward parabolic) [16] with respect to the value $C = C(S, t')$, which is a European call option, where S and t' are the current values of the underlying asset and time,

$$\frac{\partial C}{\partial t'} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\, S\, \frac{\partial C}{\partial S} - r\, C = 0, \quad (S, t') \in I\!R^+ \times [0, T), \tag{1.1a}$$

with the final condition

$$C(S, T) = \max(S - E, 0), \quad S \in I\!R^+, \tag{1.1b}$$

and the boundary conditions at $S = 0$ and at infinity $S = +\infty$

$$C(0, t') = 0; \quad C(S, t') \to S \quad \text{for} \quad S \to \infty, \quad t' \in [0, T). \tag{1.1c}$$

Here σ, E, T and r are some financial parameters (the volatility, exercise price, expiry time and the interest rate, respectively).

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For the problem (1.1), in addition to the solution itself, some of the partial derivatives of the solution are of interest (see [16], Ch. 3).

When studying this problem, a standard approach is a transformation of the equation by the changes of variables.

By the transformations

$$S = E\exp x, \quad t' = T - \tau r^{-1}, \quad C = E\,v(x,\tau) \tag{1.1d}$$

and introducing the notation $k = 2\sigma^{-2} r$, $\tau^* = rT$, we come to the following problem for the dimensionless parabolic equation in the new variables x, τ:

$$L\,v(x,\tau) \equiv \left\{ \frac{\partial^2}{\partial x^2} + (k-1)\frac{\partial}{\partial x} - k - k\frac{\partial}{\partial \tau} \right\} v(x,\tau) = 0, \tag{1.2}$$

$$(x,\,\tau) \in I\!R \times (0,\,\tau^*]$$

with the initial condition

$$v(x,0) = \varphi_v(x), \quad x \in I\!R, \tag{1.3a}$$

where

$$\varphi_v(x) = \max(\exp x - 1,\, 0), \quad x \in I\!R,$$

and with the condition at infinity

$$\left.\begin{array}{lll} v(x,\tau) \to 0 & \text{for} & x \to -\infty \\ v(x,\tau) \to \exp x & \text{for} & x \to \ \ \infty \end{array}\right\}, \quad \tau \in (0,\tau^*]. \tag{1.3b}$$

Under the condition $T, r = \mathcal{O}(1)$ and for σ taking an arbitrary value from the half-open interval $(0, \sqrt{2r})$, we come to the Cauchy problem for the singularly perturbed parabolic equation

$$L\,v(x,\tau) \equiv \Big\{ \varepsilon\frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial \tau} \Big\} v(x,\tau) = 0, \quad (x,\tau) \in I\!R \times (0,\tau^*] \tag{1.4}$$

with conditions (1.3). Here $\varepsilon = 2^{-1}\sigma^2 r^{-1}$ is a dimensionless "perturbation" parameter, $\varepsilon \in (0,1]$.

The initial function in condition (1.3a) is continuous; its first derivative in x has a discontinuity of the first kind at the point $x = 0$

$$\left[\frac{d}{dx}\varphi_v(0) \right] = 1,$$

where the jump of the derivative is defined by the relation

$$\left[\frac{d}{dx}\varphi_v(0) \right] = \lim_{x \searrow 0} \frac{d}{dx}\varphi_v(x) - \lim_{x \nearrow 0} \frac{d}{dx}\varphi_v(x).$$

The initial function and the solution itself for this problem grow (exponentially) without bound as $x \to \infty$. If the parameter $\varepsilon = 1$ then the problem (1.4), (1.3) becomes the one of reaction-diffusion type, and for $\varepsilon < 1$, it is of convection-diffusion type. For small values of the parameter ε, an interior (moving in time) layer with the typical width of $\varepsilon^{1/2}$ appears in a neighbourhood of the characteristic (of the operator $L_1 \equiv (1-\varepsilon)\dfrac{\partial}{\partial x} - 1 - \dfrac{\partial}{\partial \tau}$) passing through the point $(0,0)$.

Thus, the Cauchy problem (1.4), (1.3) is a singularly perturbed problem with different types of singularities. In the present paper we are interested in approximations to both the solution and its first order derivative in a finite subdomain that contains the singularity of the interior layer type.

1.2. Boundary value problems in bounded domains for singularly perturbed parabolic reaction-diffusion equations with a discontinuous initial condition have been considered in [2, 3, 9, 10, 12, 13]. To construct schemes that converge ε-uniformly, the method of condensing meshes (in a neighbourhood of boundary layers), and also either the fitted operator method [2, 9, 10, 12] or the method of additive splitting of a singularity [3, 13] (in a neighbourhood of the points at which the initial function is discontinuous) were applied.

In [9, 10, 12, 13], approximations to the normalized derivatives $\varepsilon\,(\partial/\partial x)\,u(x,t)$, i.e., the first order spatial derivative multiplied by the parameter ε, were considered. For this purpose, the method of additive splitting of the singularity generated by the discontinuity of the initial function was used; however, the approximation of the derivative $(\partial/\partial x)\,u(x,t)$ itself was not considered.

A boundary value problem on a segment for singularly perturbed parabolic convection-diffusion equations with a piecewise smooth initial condition has been considered in [14, 15]. In [15], by using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of the additive splitting of a singularity of the interior layer type, special difference schemes are constructed that make it possible to approximate ε-uniformly the solution of the problem on the entire set under consideration, the normalized derivative on the entire set except for the discontinuity point $(0,0)$, and the first spatial derivative on the same set but outside a small neighbourhood of the boundary layer.

In the present paper, instead of the Cauchy problem (1.4), (1.3), we consider a singularly perturbed boundary value problem for equation (1.4) with a non-smooth initial condition similar to (1.3), namely, the problem (2.2), (2.1) (see the formulation of this problem in Section 2). The technique from [15] is used for studying the problem (2.2), (2.1). Note that in a problem of the type (2.2), (2.1) considered in a finite domain, except for the interior layer, an additional singularity appears, namely, a boundary layer with the typical width of ε. The singularity of the boundary layer is more strong than that of the interior layer, which makes it difficult to construct special numerical methods suitable for the adequate description of the singularity of the interior-layer type. In contrast to [15], here conditions are defined that allow us to investigate each singularity of the problem separately. For the boundary value problem (2.2), (2.1), we construct a finite difference scheme that approximates the solution and its first order derivative in x. To construct ε-uniform approximations for the solution and its first derivative in a finite subdomain including only the interior layer singularity, it suffices to use a uniform mesh and the method of the additive splitting of the singularity of the interior layer type. The efficiency of the scheme constructed in this paper is verified with numerical experiments.

The numerical method constructed for problem (2.2), (2.1), after the transformation to the original variables S, t' and the function C (see the change (1.1d)), allows us to approximate the solution of problem (1.1) and its first derivative $(\partial/\partial\ S)\,C(S,t')$ in a finite neighbourhood of the point (E,T) (the point of discontinuity of the derivative in condition (1.1b)), including the interior layer (appearing for small values of the dimensionless quantity $\sigma^2\,r^{-1}$). Errors in the approximation of the solution and derivative (for $(S,t')\neq(E,T)$) are independent of the value $\sigma^2\,r^{-1}$; these errors (in the maximum norm) are defined only by the number of nodes in the mesh used for the numerical solution of the discrete problem.

About Contents. Formulation of the boundary value problem is given in Section 2.

Difficulties involved in the approximation of the solution and derivatives on the base of classical finite difference schemes are discussed in Section 3. *A priori* estimates used in the constructions are presented in Section 4. Classical difference approximations of the problem on uniform and piecewise uniform meshes are considered in Section 5. A difference scheme (approximating the solution and its first order derivative), which is constructed using the method of the additive splitting of the singular part of the solution generated by the discontinuity of the derivative of the initial function, is given in Section 6. In the same place, conditions are defined under which a certain singularity of the solution can be split off and investigated separately. Numerical experiments are analyzed in Section 7.

2 Problem Formulation. Aim of Research

2.1. On the set $\overline{G}$ with the boundary S,

$$\overline{G} = G \cup S, \quad G = D \times (0,\, T], \quad D = \{x :\ x \in (-d,\, d)\}, \tag{2.1}$$

we consider the Dirichlet problem for the singularly perturbed parabolic convection-diffusion equation [1]

$$L_{(2.2a)}\, u(x,t) \;=\; f(x,t), \quad (x,t) \in G, \tag{2.2a}$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S. \tag{2.2b}$$

Here $$L_{(2.2a)} \equiv \varepsilon\, a\, \frac{\partial^2}{\partial x^2} + b\, \frac{\partial}{\partial x} - c - q\, \frac{\partial}{\partial t},$$

$a,\ b,\ q\ >\ 0,\ c\ \geq\ 0$, the right-hand side $f(x,t)$ is a sufficiently smooth function on $\overline{G}$; the parameter ε takes arbitrary values in the half-open interval $(0,\, 1]$. The boundary function $\varphi(x,t)$ is sufficiently smooth on the sets $\overline{S}_0^{\,-}$, $\overline{S}_0^{\,+}$, $\overline{S}^{\,L}$ and continuous on S; the first order derivative in x of the function $\varphi(x,t)$ has a discontinuity of the first kind on the set $S^{(*)} = \{(0,\, 0)\}$, i.e.,

$$\left[\frac{\partial}{\partial x}\varphi(x,t)\right] \neq 0, \quad (x,t) \in S^{(*)}. \tag{2.2c}$$

Here
$$S_0^- = \{(x,t) :\ x \in [-d,\, 0),\ t = 0\},$$

$$S_0^+ = \{(x,t) :\ x \in (0,\, d],\ t = 0\}, \quad S_0 = \overline{S}_0^{\,-} \bigcup \overline{S}_0^{\,+},$$

S_0 and S^L are the lower and lateral parts of the boundary S, $S^{\,L} = \Gamma \times (0,\, T]$, $\Gamma = \overline{D} \setminus D$.

The solution of problem (2.2) is a function $u \in C(\overline{G}) \bigcap C^{\,2,1}(G)$ satisfying the differential equation on G and the boundary conditions on S.

For simplicity, we assume that compatibility conditions that ensure the smoothness of the solution for fixed values of ε [4] are fulfilled on the set $S_* = S_0 \bigcap \overline{S}^{\,L}$. Let $\overline{G}^{\,\delta}$ be the δ-neighbourhood of the set S_*, i.e.,

$$\overline{G}^{\,\delta} = \{(x,t) :\ r\big((x,t),\, S_*\big) \leq \delta\},$$

where $r\big((x,t),\, S_*\big)$ is the distance from the point (x,t) to the set S_*. We suppose that the following inclusion holds on the set $\overline{G}^{\,\delta}$:

$$u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^{\,\delta}), \quad l \geq 2, \quad \alpha \in (0,\, 1). \tag{2.3}$$

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[1] The notation $L_{(j.k)}$ ($m_{(j.k)}$, $M_{(j.k)}$, $G_{h(j.k)}$)) means that these operators (constants, meshes) are introduced in formula $(j.k)$.

It follows from [4] that, under the condition (2.3), the solution of the problem (for sufficiently smooth functions $f(x,t)$ on $\overline{G}$ and $\varphi(x,t)$ on $\overline{S}_0^-$, $\overline{S}_0^+$, $\overline{S}^L$) is smooth on the set

$$\overline{G}^* = \overline{G} \setminus S^{(*)}, \tag{2.4}$$

i.e., $u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^*)$. The derivative $(\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^*$, bounded on $\overline{G}^*$ for fixed values of ε and has a discontinuity on the set $S^{(*)}_{(2.2c)}$.

Under the condition

$$a = c = p = 1, \quad b = 1 - \varepsilon, \quad f(x,t) = 0, \quad (x,t) \in \overline{G}$$

the equation (2.2a) becomes the equation (1.4).

We are interested in an approximation of the solution $u(x,t)$, $(x,t) \in \overline{G}$, and of the derivative $(\partial/\partial x)u(x,t)$, $(x,t) \in \overline{G}^*$. Let us describe the behaviour of the solution and derivatives more precisely.

Let $S^L = S^l \bigcup S^r$, S^l and S^r be the left and right parts of the boundary S^L, and let

$$S^\gamma = \{(x,t) : \ x = \gamma(t), \ (x,t) \in \overline{G}\}, \quad \gamma(t) = -b\,q^{-1}\,t, \quad t \geq 0$$

be the characteristic of the reduced equation passing through the point (0, 0). When the parameter ε tends to zero, boundary and interior layers with the typical length scales ε and $\varepsilon^{1/2}$, respectively, appear in a neighbourhood of the sets S^l and S^γ; as opposed to the boundary layer, the interior layer is weak (the first order derivative in x of the interior-layer function is bounded ε-uniformly).

For simplicity, we assume that the characteristic S^γ does not meet the boundary S^l. The derivative $(\partial/\partial x)u(x,t)$ (denoted by $p(x,t)$) in a neighbourhood of the set S^l grows without bound as $\varepsilon \to 0$. It is convenient to consider the quantity $P(x,t) = \varepsilon\,(\partial/\partial x)u(x,t)$, i.e., the normalized first derivative in x, in the m-neighbourhood of the set S^l, instead of the derivative $(\partial/\partial x)u(x,t)$, because this quantity is bounded ε-uniformly. Outside a neighbourhood of the set S^l, the derivative $(\partial/\partial x)u(x,t)$ is bounded ε-uniformly. The quantity $P(x,t)$ will be called the diffusion flux (or briefly, the flux). Outside of a neighbourhood of the set S^l, the derivative $p(x,t)$ is bounded ε-uniformly on $\overline{G}^*$. For small values of the parameter ε, the derivative $p(x,t)$ is more "informative" (on the set where it is bounded) than the flux $P(x,t)$.

2.2. It is well known (see, e.g., [1]) that even in the case of singularly perturbed problems with sufficiently smooth data, solutions of classical finite difference schemes do not converge ε-uniformly; for small values of the parameter ε, errors in the discrete solutions are commensurable with the actual solutions of the differential problem. The diffusion fluxes obtained on the basis of such schemes also do not converge ε-uniformly. It will be shown in Section 3 that for a boundary value problem whose solution is regular, classical difference schemes do not allow one to obtain ε-uniformly convergent approximations of the derivative in x.

Due to this it would be interesting to construct a difference scheme that allows us to approximate ε-uniformly both the solution on the whole domain $\overline{G}$ and diffusion fluxes in this domain excluding the discontinuity point $S^{(*)}$. Also, it will be interesting to determine conditions under which the boundary layer does not appear, and for such a problem, to find the ε-uniform approximation of the derivative in x on the set $\overline{G}^*$.

Definition. Let

$$\overline{G}_0^* = \overline{G}_0^*(m) = \overline{G}^* \cap \{x \geq -d + m\} \tag{2.5}$$

be the set $\overline{G}^*$ excluding an m-neighbourhood[2] of the set $\overline{S}^l$ (the m-neighbourhood of the boundary layer). If the interpolants constructed using the solution of some finite difference scheme converge on $\overline{G}$ ε-uniformly, we say that the discrete solution $\big($the difference scheme$\big)$ converges on $\overline{G}$ uniformly with respect to the parameter ε (or, briefly, ε-uniformly) in $C(\overline{G})\big)$. If, moreover, the interpolants of the diffusion fluxes (the first order derivatives in x) converge ε-uniformly on $\overline{G}$ $\big(\varepsilon$-uniformly on $\overline{G}_0^*\big)$, we say that the difference scheme converges ε-uniformly in $C^{\,1(n)}(\overline{G}^*)$ $\big(\varepsilon$-uniformly in $C^{\,1}(\overline{G}_0^*)\big)$.

Thus, it is attractive to find numerical methods that converge ε-uniformly in $C^{\,1(n)}(\overline{G}^*) \cap C^{\,1}(\overline{G}_0^*)$, where $\overline{G}_0^* = \overline{G}_0^*(m)$, moreover, it is required that the value m could be chosen sufficiently small.

Our aim is to construct a difference scheme for problem (2.2), (2.1) that converges ε-uniformly in $C^{\,1(n)}(\overline{G}^*) \cap C^{\,1}(\overline{G}_0^*)$, and also to determine conditions under which the boundary layer does not appear, and in this case to construct a difference scheme that converges ε-uniformly in $C^{\,1}(\overline{G}^*)$.

Some preliminary results related to this problem are given in [5, 6]. To investigate the problem, a technique similar to that developed in [15] is used. In the present paper, in contrast to [15], the main attention is given to the study of a singularity of the interior-layer type, because the boundary layer does not arise in the original problem (1.1).

3 On the approximation of the derivative in x

3.1. Let us discuss difficulties arising in the approximation of derivatives for the regular components of the problem solution, i.e., when the solution of a singularly perturbed problem is regular (not containing the singular component) and sufficiently smooth. In this case the solution of a classical finite difference scheme on a uniform mesh converges to the exact solution ε-uniformly. However, its difference derivatives are no longer convergent ε-uniformly; the error in the derivative of the solution of the grid problem can have the order of the derivative of the solution itself for the differential problem.

Consider the stationary problem

$$\begin{aligned} L_{(3.1)}\, u(x) \;&\equiv\; \left\{\varepsilon \frac{d^2}{dx^2} + \frac{d}{dx}\right\} u(x) = f(x), \quad x \in D, \\ u(x) \;&=\; \varphi(x), \quad x \in \Gamma. \end{aligned} \tag{3.1a}$$

Here

$$\overline{D} = [0,\, 1]. \tag{3.1b}$$

Let $u(x) = x^2$, $x \in \overline{D}$, be a solution of problem (3.1); this solution has no singular component.

To solve problem (3.1), we apply the classical difference scheme [8]. On the segment $\overline{D}$, we introduce the uniform mesh

$$\overline{D}_h = \overline{D}_{h(3.2)} \tag{3.2}$$

with the step-size $h = N^{-1}$. On the mesh $\overline{D}_h$, we consider the difference scheme

$$\begin{aligned} \Lambda_{(3.3)}\, z(x) \;&\equiv\; \{\varepsilon\, \delta_{x\overline{x}} + \delta_x\}\; z(x) = f(x), \quad x \in D_h, \\ z(x) \;&=\; \varphi(x), \quad x \in \Gamma_h. \end{aligned} \tag{3.3}$$

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[2] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

The solution of problem (3.3), (3.2) has the explicit form

$$z(x_i) = x_i^2 + N^{-1}\big\{1 - x_i - [(1+\varepsilon^{-1}N^{-1})^{-i} - (1+\varepsilon^{-1}N^{-1})^{-N}] \times$$
$$\times [1 - (1+\varepsilon^{-1}N^{-1})^{-N}]^{-1}\big\}, \quad x_i = i\,N^{-1} \in \overline{D}_h, \quad i \le N.$$

Thus, the function $z(x)$, $x \in \overline{D}_h$, converges to $u(x)$, $x \in \overline{D}$, ε-uniformly with the estimate

$$|\, u(x) - z(x) \,| \le N^{-1}, \quad x \in \overline{D}_h.$$

For the first discrete derivative

$$\delta_x\, z(x_i) = 2\,x_i + N^{-1}(\varepsilon + N^{-1})^{-1}(1+\varepsilon^{-1}N^{-1})^{-i} \times [1 - (1+\varepsilon^{-1}N^{-1})^{-N}]^{-1},$$
$$x_i = i\,N^{-1} \in \overline{D}_h, \quad i \le N-1$$

we have the error

$$\left|\frac{d}{dx}\,u(x_i) - \delta_x\, z(x_i)\right| = N^{-1}(\varepsilon + N^{-1})^{-1}(1+\varepsilon^{-1}N^{-1})^{-i} \times$$
$$\times [1 - (1+\varepsilon^{-1}N^{-1})^{-N}]^{-1}, \quad x_i \in \overline{D}_h, \quad x_i < 1;$$
$$\max_{\overline{D}_h,\ x<1}\left|\frac{d}{dx}\,u(x) - \delta_x\, z(x)\right| \ge m\,N^{-1}(\varepsilon + N^{-1})^{-1}.$$

Thus, the discrete derivative does not converge ε-uniformly: when $\varepsilon \le N^{-1}$, the error in this discrete derivative is of the order of the derivative itself.

3.2. If the solution of the boundary value problem (2.2), (2.1) is regular, moreover, the regular component is of the order of unity in a neighbourhood of the boundary layer, then the error in the approximation of the derivative $(\partial/\partial x)u(x,t)$, in general, grows unboundedly under the condition $(\varepsilon + N^{-1})^{-1}N_0^{-1} \to \infty$ as N, $N_0 \to \infty$, where $N+1$ and N_0+1 are the number of nodes with respect to x and t in the uniform mesh on $\overline{G}$.

4 *A priori* estimates of the solution and derivatives

For the solution of the boundary value problem (2.2), (2.1) and its derivatives, we give *a priori* estimates used in the constructions $\big($the more detailed derivation can be found in [15]$\big)$.

4.1. We represent the set $\overline{G}$ as the sum of overlapping sets

$$\overline{G} = \bigcup\nolimits_j \overline{G}^{\,j}, \quad j = 1,\, 2,\, 3, \tag{4.1}$$

where

$$G^{\,1} = G^{\,1}(m^1) = \{(x,t):\ |x - \gamma(t)| < m^1, \ \ t \in (0,T]\},$$
$$G^{\,2} = G^{\,2}(m^2) = \{(x,t):\ x \in (-d, -d + m^2), \ \ t \in (0,T]\},$$
$$G^{\,3} = G^{\,3}(m^3) = G \setminus \left\{G^1(m^3) \bigcup G^2(m^3)\right\}, \quad m^3 < m^1,\, m^2,$$

$G^{\,1}$ and $G^{\,2}$ are neighbourhoods of the interior and boundary layers, respectively. The solution of problem (2.2), (2.1) considered on the set $\overline{G}^{\,j}$ will be also denoted by the $u^j(x,t)$, $j = 1,\, 2,\, 3$.

The solution on the set $\overline{G}^{\,3}$ is smooth; we have the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u^3(x,t) \right| \le M, \quad (x,t) \in \overline{G}^{\,3}, \quad k + 2k_0 \le K. \tag{4.2}$$

The value K is defined by the problem data; and $K \ge 4$.

We represent the solution on the set $\overline{G}^{\,2}$ as the decomposition into two functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}^{\,2}, \tag{4.3}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution. For the functions $U(x,t)$ and $V(x,t)$, the following estimates are valid:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,\, t) \right| \le M, \tag{4.4a}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,\, t) \right| \le M\, \varepsilon^{-k} \exp\Big(-m\, \varepsilon^{-1}\, r\big((x,t),\, \overline{S}^{l}\big)\Big), \tag{4.4b}$$

$$(x,t) \in \overline{G}^{2}, \quad k + 2\, k_0 \le K,$$

where $r\big((x,t),\, \overline{S}^{\,l}\big)$ is the distance from the point (x,t) to the set $\overline{S}^{\,l}$, m is an arbitrary constant from the interval $(0,\, m_0)$, $m_0 = a^{-1}\, b$.

4.2. On the set $\overline{G}^{\,1}$, we introduce the new variables

$$\widetilde{u}(\xi,\, t) = u\big(x(\xi, t),\, t\big)\, \exp(\alpha\, t), \quad (\xi,\, t) \in \overline{\widetilde{G}}^{\,1}, \quad \xi = x - \gamma(t), \quad (x,\, t) \in \overline{G}^{\,1}. \tag{4.5}$$

Here $\gamma(t) = -b\, q^{-1}\, t$, $\alpha = c\, q^{-1}$, and $\widetilde{G}^{\,1}$ is the image of the set G^1. The function $\widetilde{u}(\xi,\, t)$, $(\xi,\, t) \in \widetilde{G}^{\,1}$, is the solution of the problem

$$L_{(4.6a)} \widetilde{u}(\xi,\, t) \;\equiv\; \left\{ \varepsilon\, a\, \frac{\partial^2}{\partial \xi^2} - q\, \frac{\partial}{\partial t} \right\} \widetilde{u}(\xi,\, t) = \widetilde{f}(\xi,\, t), \quad (\xi,\, t) \in \widetilde{G}^{\,1}, \tag{4.6a}$$

$$\widetilde{u}(\xi,\, t) \;=\; \begin{cases} \widetilde{u}^{\,3}(\xi,\, t), & (\xi,\, t) \in \widetilde{S}^{\,1} \setminus \widetilde{S}, \\ \widetilde{\varphi}(\xi,\, t), & (\xi,\, t) \in \widetilde{S}^{\,1} \bigcap \widetilde{S}. \end{cases}$$

Here $\widetilde{S}^{\,1} = \overline{\widetilde{G}}^{\,1} \setminus \widetilde{G}^{\,1}$, and $\widetilde{v}(\xi,\, t)$ is the image of the function $v(x,t)$,

$$\widetilde{v}(\xi,\, t) = v\big(x(\xi,\, t),\, t\big)\, \exp(\alpha\, t), \tag{4.6b}$$

where $v(x,t)$ is one of the functions $u(x,t)$, $f(x,t)$, $(x,\, t) \in \overline{G}^{\,1}$, $\varphi(x,\, t)$, $(x,\, t) \in S^{\,1} \bigcap \{t = 0\}$, or $u^{\,3}(x,t)$, $(x,\, t) \in \overline{G}^{\,1} \bigcap \overline{G}^{\,3}$, $u^{\,3}(x,t) = u(x,t)$, $(x,\, t) \in \overline{G}^{\,3}$.

The function $\widetilde{u}(\xi,\, t)$ is decomposed into the sum of functions

$$\widetilde{u}(\xi,\, t) = \widetilde{U}^{\,1}(\xi,\, t) + \widetilde{W}^{\,1}(\xi,\, t), \quad (\xi,\, t) \in \overline{\widetilde{G}}^{\,1}, \tag{4.7a}$$

where $\widetilde{U}^{\,1}(\xi,t)$ and $\widetilde{W}^{\,1}(\xi,t)$ are the regular ("sufficiently" smooth) and singular parts of the solution. The function $\widetilde{W}^{\,1}(\xi,t)$ is the solution of the Cauchy problem

$$L_{(4.6)}\, \widetilde{W}^{\,1}(\xi,\, t) \;=\; 0, \qquad (\xi,t) \in I\!R \times (0,\, T], \tag{4.8}$$

$$\widetilde{W}^{\,1}(\xi,\, t) \;=\; \widetilde{\Phi}^{1}_{W}(\xi), \quad \xi \in I\!R, \quad t = 0,$$

Here

$$\widetilde{\Phi}^1_W(\xi) = 2^{-1} \left[\frac{\partial}{\partial \xi}\,\widetilde{\varphi}(0,\,0)\right] |\xi|, \quad \xi \in I\!R,$$

$\left[\frac{\partial}{\partial \xi}\,\widetilde{\varphi}(0,\,0)\right] = \frac{\partial}{\partial \xi}\,\widetilde{\varphi}(+0,\,0) - \frac{\partial}{\partial \xi}\,\widetilde{\varphi}(-0,\,0)$ is the jump of the derivative $\left[\frac{\partial}{\partial \xi}\,\widetilde{\varphi}(\xi,\,t)\right]$. The function $\widetilde{W}^{\,1}(\xi,\,t)$ takes the form

$$\widetilde{W}^{\,1}(\xi,\ t) = 2^{-1} \left[\frac{\partial}{\partial \xi}\,\widetilde{\varphi}(0,\,0)\right] \widetilde{w}_1(\xi,\,t), \tag{4.7b}$$

$$\widetilde{w}_1(\xi,\,t) = \xi\, v\big(2^{-1}\,\varepsilon^{-1/2}\,a^{-1/2}\,q^{1/2}\,\xi\,t^{-1/2}\big) +$$
$$+2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp\big(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\xi^2\,t^{-1}\big), \quad (\xi, t) \in I\!R \times [0,\,T],$$
$$v(\xi) = erf(\xi) = 2\,\pi^{-1/2} \int_0^{\xi} \exp(-\alpha^2)\,d\alpha, \quad \xi \in I\!R.$$

The function $\widetilde{U}^{\,1}(\xi,\,t)$ is the solution of the problem

$$\begin{aligned} L_{(4.6)}\,\widetilde{U}^{\,1}(\xi,\,t) &= \widetilde{f}(\xi,\,t), \quad (\xi,\,t) \in \widetilde{G}^{\,1}, \\ \widetilde{U}^{\,1}(\xi,\,t) &= \begin{cases} \widetilde{u}^{\,3}(\xi,\,t) - \widetilde{W}^{\,1}(\xi,\,t) & (\xi,\,t) \in \widetilde{S}^{\,1}, \quad t > 0, \\ \widetilde{\varphi}(\xi,\,t) - \widetilde{\Phi}^1_W(\xi,\,t) & (\xi,\,t) \in \widetilde{S}^{\,1}, \quad t = 0. \end{cases} \end{aligned}$$

Note that the first derivative of the function $\widetilde{U}^{\,1}(\xi,\,t)$ in ξ is continuous at $t = 0$.

4.3. In the variables $x,\ t$, the following representation corresponds to representation (4.7a):

$$u(x,t) = U^1(x,t) + W^1(x,t), \quad (x,\,t) \in \overline{G}^{\,1}, \tag{4.9a}$$

where

$$W^1(x,t) = \widetilde{W}^{\,1}\big(\xi(x,t),\,t\big)\,\exp(-\alpha\,t) = 2^{-1}\left[\frac{\partial}{\partial x}\varphi(0,\,0)\right] \times \tag{4.9b}$$
$$\times\Big\{\big(x-\gamma(t)\big)\,v\Big(2^{-1}\,\varepsilon^{-1/2}\,a^{-1/2}\,q^{1/2}\,\big(x-\gamma(t)\big)t^{-1/2}\Big) +$$
$$+2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp\Big(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\big(x-\gamma(t)\big)^2\,t^{-1}\Big)\Big\} \times$$
$$\times \exp(-\alpha\,t), \quad (x,t) \in I\!R \times [0,\,T].$$

For the components in representation (4.9), we have the estimates (similar to those in [15])

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U^1(x,t) \right| \le M \left[1 + \varepsilon^{(2-k-k_0)/2}\, \rho^{2-k-k_0} + \varepsilon^{(2-k)/2}\, \rho^{2-k-2k_0} \right],$$

$$(x,t) \in \overline{G}^{\,1},$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, W^1(x,t) \right| \le M \left[1 + \varepsilon^{(1-k-k_0)/2}\, \rho^{1-k-k_0} + \varepsilon^{(1-k)/2}\, \rho^{1-k-2k_0} \right] \times$$

$$\times \exp(-m\, \varepsilon^{-1/2}\, |x - \gamma(t)|\,), \quad (x,t) \in \overline{G}; \tag{4.10}$$

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}} \widetilde{U}^1(\xi,t) \right| \le M \big[1 + \varepsilon^{(2-k)/2}\, \widetilde{\rho}^{\,2-k-2\,k_0} \big], \qquad (\xi,t) \in \overline{\widetilde{G}}^{\,1},$$

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}} \widetilde{W}^1(\xi,t) \right| \le M \big[1 + \varepsilon^{(1-k)/2}\, \widetilde{\rho}^{\,1-k-2\,k_0} \big]\, \exp(-m\, \varepsilon^{-1/2}\, |\xi|\,),$$

$$(\xi,t) \in \overline{\widetilde{G}}, \quad k + 2k_0 \le\, K,$$

where $\rho = \rho(x,t;\,\varepsilon) = \varepsilon^{-1/2} \left| x - \gamma(t) \right| + t^{1/2}$, $\widetilde{\rho} = \widetilde{\rho}(\xi,t;\,\varepsilon) = \varepsilon^{-1/2} \left| \xi \right| + t^{1/2}$, and m is an arbitrary constant.

Thus, the following theorem is valid.

Theorem 4.1 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the condition* $f \in C^{\,l,l/2}(\overline{G})$, $\varphi \in C^{\,l}(\overline{S}_0^{\,-}) \bigcap C^{\,l}(\overline{S}_0^{\,+}) \bigcap C^{\,l/2}(\overline{S}^{\,L}) \bigcap C(S)$, *and let the condition* (2.3) *holds for the solution of this problem, moreover,* $l = K$. *Then the solution of the boundary value problem and its components in representations* (4.3), (4.9) *satisfy the estimates* (4.2), (4.4), *and* (4.10).

Remark 4.1 Let us give the condition under which the boundary layer does not arise.

Let us define the set

$$G^4 = G^4(m) = \big\{ (x,t);\, x \,>\, \gamma(t) - \gamma\,(T) - d + m \big\}, \quad \overline{G}^{\,4} = G^{\,4} \bigcup S^{\,4}, \tag{4.11a}$$

where $m \,<\, d + \gamma\,(T)$. Introduce the set

$$\overline{G}^{\,5} = G^{\,5} \bigcup S^{\,5}, \quad G^{\,5} = G^{\,5}(m) = G \setminus \overline{G}^{\,4}(m). \tag{4.11b}$$

Let the functions $f(x,t)$ and $\varphi(x,t)$ satisfy the conditions

$$\begin{aligned} f(x,t) \;&=\; 0, \quad (x,\,t) \in \overline{G}^{\,5}, \\ \varphi(x,t) \;&=\; 0, \quad (x,\,t) \in\, S \bigcap \overline{G}^{\,5}. \end{aligned} \tag{4.12}$$

Then the boundary layer is absent, i.e., the singular component is absent in the representation (4.3):

$$V(x,t) = 0, \quad (x,\,t) \in \overline{G}^{\,2}, \tag{4.13}$$

and $u(x,t) = U(x,t)$ on the set $\overline{G}^{\,2}$. For the solution $u(x,t)$, the estimate (4.4a) takes place, moreover,

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \le\; M\, \varepsilon^{K_1}, \quad (x,t) \in \overline{G}^{\,2}, \quad k + 2k_0 \le K, \tag{4.14}$$

where $\overline{G}^{\,2} = \overline{G}^{\,2}_{(4.1)}(m^2)$, $m^2 < m_{(4.1)}$, and the constant K_1 in (4.14) can be chosen sufficiently large.

■

5 Classical grid approximations of the problem on uniform and piecewise uniform meshes

5.1. Let us consider a difference scheme based on classical approximations of problem (2.2), (2.1). On the set $\overline{G}_{(2.1)}$, we introduce the rectangular mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{5.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the segments $[-d,\, d]$ and $[0,\, T]$, respectively; the mesh $\overline{\omega}$ has an arbitrary distribution of nodes satisfying only the condition $h \le MN^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$; the mesh $\overline{\omega}_0$ is uniform with the step-size $h_0 = TN_0^{-1}$. Here $N+1$ and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

We approximate the boundary value problem (2.2) by the difference scheme

$$\begin{aligned} \Lambda_{(5.2)}\, z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.2}$$

Here

$$\Lambda_{(5.2)} \equiv \varepsilon\, a\, \delta_{\overline{x}\,\widehat{x}} + b\, \delta_x - c - q\, \delta_{\overline{t}},$$

$\delta_{\overline{x}\,\widehat{x}}\, z(x,t)$ is the second difference derivative on a nonuniform mesh. On the uniform mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{5.3}$$

we obtain the estimate

$$|u(x,t) - z(x,t)| \le M \left[\varepsilon^{-1} N^{-1} + N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h, \tag{5.4}$$

i.e., the scheme (5.2), (5.3) converges for fixed values of the parameter ε.

Under the condition (4.13), when $V(x,t) = 0$, $(x,t) \in \overline{G}^{\,2}$, we have the estimate

$$|u(x,t) - z(x,t)| \le M \left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h, \tag{5.5}$$

i.e., the scheme (5.2), (5.3) converges ε-uniformly.

If, except (4.13), the following condition holds:

$$W^1(x,t) = 0, \quad (x,t) \in \overline{G}^{\,1}, \tag{5.6}$$

i.e.,

$$\left[\frac{\partial}{\partial x}\, \varphi(x,t)\right] = 0, \qquad (x,t) \in S^{(*)},$$

then we have the ε-uniform estimate for the solution of difference scheme (5.2), (5.3):

$$|u(x,t) - z(x,t)| \le M \left[N^{-1} + N_0^{-1+\nu_0}\right], \quad (x,t) \in \overline{G}_h, \tag{5.7}$$

where ν_0 is an arbitrary constant in the interval $(0,\, 1)$.

The following theorem holds.

Theorem 5.1 *Let the solution of problem* (2.2) *and its components satisfy the estimates* (4.2), (4.4), (4.8) *for* $K = 4$. *Then the difference scheme* (5.2), (5.3) *converges for fixed values of the parameter* ε. *In the case of the condition* (4.13), *the scheme* (5.2), (5.3) *converges* ε*-uniformly. The discrete solutions satisfy the estimate* (5.4)*; and, in the case of the condition* (4.13), *the estimate* (5.5) *holds, and under conditions* (4.13) *and* (5.6), *the estimate* (5.7) *is valid.*

5.2. We now consider the case when the solution of the problem has a boundary layer. On the set $\overline{G}$, we construct the mesh condensing in a neighbourhood of the boundary layer (similar to that constructed in [1, 7, 11, 15]):

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^* \times \overline{\omega}_0, \tag{5.8a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(5.1)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $[-d,\, d]$, and σ is a parameter depending on ε and N. We choose the value σ satisfying the condition

$$\sigma = \sigma(\varepsilon,\, N) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\,\ln N\,], \tag{5.8b}$$

where β is an arbitrary number in the half-open interval $(0,\, d]$ and $m = m_{(4.4)}$. The segment $[-d,\, d]$ is divided into two parts: $[-d,\, -d+\sigma]$ and $[-d+\sigma,\, d]$; on each part, the step-size is constant and is equal to $h^{(1)} = 2\,d\,\sigma\,\beta^{-1}\,N^{-1}$ on the segment $[-d, -d+\sigma]$ and to $h^{(2)} = 2\,d\,(2\,d-\sigma)\,(2\,d-\beta)^{-1}\,N^{-1}$ on the segment $[-d+\sigma, d]$, $\sigma \le d$.

On the mesh (5.8), we have the estimate

$$\mid u(x,t) - z(x,t) \mid \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h, \tag{5.9}$$

i.e., the scheme (5.2), (5.8) converges ε-uniformly.

The following theorem holds [11].

Theorem 5.2 *Let the assumptions of Theorem* 5.1 *be fulfilled. Then the difference scheme* (5.2)*,* (5.8) *converges ε-uniformly with the estimate* (5.9)*.*

5.3. Let the conditions (4.13), (5.6) be satisfied (i.e., the interior and boundary layers are absent).

On the uniform mesh (5.3), the discrete derivative $p^h(x,t)$ is the solution of the equation

$$\Lambda_{(5.2)}\, p^h(x,t) = \delta_x\, f(x,t), \quad (x,t) \in G_h, \quad x \le d - 2\,h;$$

on the set S_{0h}, the following condition holds:

$$p^h(x,t) = \delta_x\, \varphi(x,t), \quad (x,t) \in S_{0h}, \quad x \ne d.$$

The following estimates are satisfied for the flux and for the discrete derivative:

$$\mid P(x,t) - P^h(x,t) \mid \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h, \quad x \ne d; \tag{5.10a}$$

$$\begin{aligned} &\mid p(x,t) - p^h(x,t) \mid \le \\ &\quad \le\; M\,[\,N^{-1/2} + N_0^{-1/2}\,]\,[(\varepsilon + N^{-1})^{-1}(1 + 2\,a^{-1}\,b\,d\,\varepsilon^{-1}\,N^{-1})^{-i} + 1], \\ &\qquad\qquad (x,t) \in \overline{G}_h, \quad x = x_i \ne d; \end{aligned} \tag{5.10b}$$

$$\begin{aligned} &\mid p(x,t) - p^h(x,t) \mid \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \\ &\qquad\qquad (x,t) \in \overline{G}_h, \quad x \in [-d + \beta_0,\, d). \end{aligned} \tag{5.11}$$

Here $\beta_0 > 0$ is an arbitrary sufficiently small constant, and $M_{(5.11)} = M(\beta_0)$.

But if the condition (4.12) is fulfilled (which implies the condition (4.13)), and also the condition (5.6) holds, then for the discrete derivative $p(x,\, t)$, we obtain the estimate

$$\left| p(x,\, t) - p^h(x,\, t) \right| \le M\,\left[N^{-1/2} + N_0^{-1/2} \right], \quad (x,\, t) \in \overline{G}_h, \quad x \ne d, \tag{5.12}$$

which is stronger than (5.11). Thus, in the case of scheme (5.2), (5.3), the condition (4.12) is sufficient for the ε-uniform convergence of the derivative $p^h(x,\, t)$ on the whole set $\overline{G}_h$, $x \neq d$.

5.4. We consider the approximation of the functions $u(x,t)$, $p(x,t)$, $P(x,t)$, $(x,t) \in \overline{G}$, using the interpolants constructed on the basis of the functions $z(x,t)$, $p^h(x,t)$, $P^h(x,t)$.

Let $z(x,t)$, $(x,t) \in \overline{G}_h$, be a solution of some scheme. For the function $z(x,t)$, we construct its extension $\overline{z}(x,t)$ to $\overline{G}$; $\overline{z}(x,t)$ is a bilinear interpolant on the elementary rectangles generated by the lines that pass through the nodes of the mesh $\overline{G}_h$ in parallel to the coordinate axes. Further, we construct the interpolant $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$, for the discrete derivative $p^h(x,t)$, $(x,t) \in \overline{G}_h$, $x \neq d$. At the interior points of the elementary rectangles, we assume $\overline{p}^h(x,t) = \overline{p}^h_z(x,t) = (\partial/\partial x)\,\overline{z}(x,t)$; the function $\overline{p}^h(x,t)$ is continuous on the upper and on the lower sides of the rectangles, and it is defined according to continuity on the left sides of the elementary rectangles. But if the rectangles are adjacent, by their right sides, to the set $\overline{S}^{\,r}$ (where $S^{\,r}$ is the right side of the boundary $S^{\,L}$, $S^{\,L} = S^{\,l} \bigcup S^{\,r}$), then we also define according to continuity the function $\overline{p}^h(x,t)$ on these sides. Hence, we have constructed the function $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$. The interpolant $\overline{p}^h(x,t)$, in general, has discontinuities on the lines that are parallel to the t-axis and pass through the nodes of the mesh G_h. We define the interpolant of the diffusion flux by the relation $\overline{P}^h(x,t) = \overline{P}^h_z(x,t) = \varepsilon\,\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$.

In the case of the difference scheme (5.2), (5.3) under conditions (4.13) and (5.6), we have the following estimates for the interpolants:

$$| \, p(x,t) - \overline{p}^{\,h}(x,t) \, | \leq M \, [\, N^{-1/2} + N_0^{-1/2} \,], \tag{5.13a}$$

$$(x,t) \in \overline{G}, \quad x \geq -d + \beta_0;$$

$$| \, P(x,t) - \overline{P}^{\,h}(x,t) \, | \leq M \, [\, N^{-1/2} + N_0^{-1/2} \,], \quad (x,t) \in \overline{G}, \tag{5.13b}$$

where $\beta_0 = \beta_{0(5.11)}$, $M_{(5.13a)} = M(\beta_0)$, and also the estimate similar to (5.7):

$$| \, u(x,t) - \overline{z}(x,t) \, | \leq M \, [\, N^{-1} + N_0^{-1+\nu_0} \,], \quad (x,t) \in \overline{G}, \tag{5.13c}$$

where $\nu_0 = \nu_{0(5.7)}$.

Definition. In that case when the interpolants constructed on the basis of the solution of the difference scheme approximate the solution of the differential problem, its derivative and the diffusion flux ε-uniformly, we say that the difference scheme approximates the solution of the differential problem, its derivative and the diffusion flux ε-uniformly.

Theorem 5.3 *Let the assumptions of Theorem* 5.1 *be fulfilled for* $K = 6$. *Then, under the conditions* (4.13), (5.6), *the difference scheme* (5.2), (5.3) *approximates the solution of problem* (2.2), (2.1), *its derivative and the diffusion flux ε-uniformly with the estimates* (5.13); *for the diffusion flux and for the derivative, the estimates* (5.10), (5.11) *are also valid.*

6 Scheme of the decomposition method for the solution approximating the derivative $p(x,t)$

To construct a difference scheme that approximates the first order derivative $p(x,t) = (\partial/\partial x)u(x,t)$ on the set $\overline{G}^{\,*}$, we use the method of the additive splitting of a singularity such as the interior layer function [15] (or briefly, the singularity splitting method).

6.1. We represent the solution of problem (2.2), (2.1) as the sum of functions

$$u(x,t) = u_1(x,t) + u_2(x,t), \quad (x,\, t) \in \overline{G}. \tag{6.1a}$$

Here $u_1(x,t)$ and $u_2(x,t)$ are components of the solution of boundary value problem (2.2), (2.1), including singularities of the boundary and interior layers types, respectively. We call the functions $u_1(x,t)$ and $u_2(x,t)$ the components containing the boundary and interior layers, respectively. We represent the function $u_2(x,t)$ as the sum of functions

$$u_2(x,t) = u_2^1(x,t) + u_2^2(x,t), \quad (x,\, t) \in \overline{G}, \tag{6.1b}$$

where $u_2^1(x,t)$ and $u_2^2(x,t)$ are the regular and singular parts of the component $u_2(x,t)$, containing the boundary layer;

$$u_2^2(x,t) = W^1_{(4.9\mathrm{b})}(x,t), \quad (x,\, t) \in \overline{G}.$$

The functions $u_1(x,t)$, $u_2^1(x,t)$ are solutions of the following problems

$$\begin{aligned} L\, u_2^1(x,t) &= f_2(x,t), \quad (x,\, t) \in G, && (6.1\mathrm{c})\\ u_2^1(x,t) &= \varphi_2(x,t), \quad (x,\, t) \in S; &&\\ L\, u_1(x,t) &= f_1(x,t), \quad (x,\, t) \in G, && (6.1\mathrm{d})\\ u_1(x,t) &= \varphi_1(x,t), \quad (x,\, t) \in S. && \end{aligned}$$

The functions $f_i(x,t)$, $\varphi_i(x,t)$, $i = 1,\, 2$ are defined by the relations

$$\begin{aligned} & f_2(x,t) = f(x,t)\,\eta\,(x,t), && (6.1\mathrm{e})\\ & f_1(x,t) = f(x,t) - f_2(x,t), \quad (x,t) \in \overline{G}; &&\\ & \varphi_2(x,t) = (\varphi(x,t) - u_2^2(x,t))\,\eta\,(x,t), &&\\ & \varphi_1(x,t) = \varphi(x,t) - \varphi_2(x,t) - u_2^2(x,t), \quad (x,t) \in S. && \end{aligned}$$

Here $\eta\,(x,t)$, $(x,t) \in \overline{G}$ is a sufficiently smooth function, that vanishes in a neighbourhood of the boundary layer

$$\left.\begin{aligned} &\eta\,(x,t) = 0, \quad (x,t) \in \overline{G}^{\,5}_{(4.11)}(2^{-1}\,m_1)\\ &\eta\,(x,t) = 1, \quad (x,t) \in \overline{G}^{\,4}_{(4.11)}(m_1) \end{aligned}\right\}, \quad 0 \le \eta(x,t) \le 1, \quad (x,t) \in \overline{G},$$

where m_1 is an arbitrary number in the interval $\Big(0,\, 2^{-1}\,\big(d + \gamma(T)\big)\Big)$.

Remark 6.1 The data of problem (6.1c) on the set $\overline{G}^{\,5}$ satisfy the condition similar to (4.12) $\big(f_2(x,t) = 0,\ \varphi_2(x,t) = 0,\ (x,t) \in \overline{G}^{\,5}\big)$, moreover, for $t = 0$, the first order derivative in x of the function $\varphi_2(x,t)$ is continuous. For the singular components of the solution to problem (6.1c) in representations similar to (4.3), (4.9a), conditions of the type (4.13), (5.6) are satisfied. For this problem, a difference scheme, similar to (5.2), on the uniform mesh (5.3) ensures the ε-uniform convergence in $C^1(\overline{G})$.

The data of problem (6.1d) are sufficiently smooth, moreover, the functions $f_1(x,t)$ and $\varphi_1(x,t)$ vanish on the set $\overline{G}^{\,4}(m_1)$. For this problem, a difference scheme, similar to (5.2), on the piecewise uniform mesh (5.8) gives the ε-uniform convergence in $C^{1(n)}(\overline{G})$. ■

To solve problem (6.1d), we use the difference scheme

$$\begin{aligned} \Lambda_{(5.2)} z_1(x,\,t) &= f_1(x,t), \quad (x,t) \in G_h, \\ z_1(x,t) &= \varphi_1(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{6.2a}$$

where $\overline{G}_h$ is the piecewise uniform mesh (5.8).

To solve problem (6.1c), we use the difference scheme

$$\begin{aligned} \Lambda_{(5.2)} z_2^1(x,\,t) &= f_2(x,t), \quad (x,t) \in G_h, \\ z_2^1(x,t) &= \varphi_2(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{6.2b}$$

where $\overline{G}_h$ is the uniform mesh (5.3).

Further, we construct the special interpolants into which the singular component, i.e., the function of the interior layer type, enters in the explicit form

$$\begin{aligned} u_0^h(x,t) &= \overline{z}_1(x,t) + u_2^h(x,t), \\ u_2^h(x,t) &= \overline{z}_2^1(x,t) + u_2^2(x,t), \quad (x,t) \in \overline{G}; \end{aligned} \tag{6.2c}$$

$$\begin{aligned} p_0^h(x,t) &= \overline{p}^{\,h}_{z_1}(x,t) + p_2^h(x,t), \\ p_2^h(x,t) &= \overline{p}^{\,h}_{z_2^1}(x,t) + \frac{\partial}{\partial x}\, u_2^2(x,t), \quad (x,t) \in \overline{G}^{\,*}; \end{aligned} \tag{6.2d}$$

$$P_0^h(x,t) = \varepsilon\, p_0^h(x,t), \quad (x,t) \in \overline{G}^{\,*}, \tag{6.2e}$$

where $\overline{z}_1(x,t)$, $\overline{p}^{\,h}_{z_1}(x,t)$ and $\overline{z}_2(x,t)$, $\overline{p}^{\,h}_{z_2}(x,t)$ are bilinear interpolants that are constructed using the functions $z_1(x,t)$, $(x,t) \in \overline{G}_{h(5.8)}$ and $z_2(x,t)$, $(x,t) \in \overline{G}_{h(5.3)}$ $\big($similarly to the construction of interpolants in Subsection 5.4$\big)$. The use of the interpolants allows us to find the solution on the set $\overline{G}$, its first derivative in x and the diffusion flux on the set $\overline{G}^{\,*}$.

The function $u_0^h(x,t)$, $(x,t) \in \overline{G}$, is called the solution of the difference scheme (6.2), (5.3), (5.8), and the functions $p_0^h(x,t)$ and $P_0^h(x,t)$, $(x,t) \in \overline{G}^{\,*}$, are called the derivative and the diffusion flux, respectively, corresponding to this scheme. The scheme (6.2), (5.3), (5.8) is the scheme of the decomposition method for the solution in the case of the additive splitting of a singularity of the interior-layer type $\big($briefly, we call this scheme by the scheme of the singularity splitting method$\big)$.

6.2. If the condition (4.12) or the following (stronger) condition are fulfilled:

$$\begin{aligned} f(x,t) &= 0, \quad (x,t) \in \overline{G}; \\ \varphi(x,t) &= 0, \quad (x,t) \in S, \quad x\,<\,0, \end{aligned} \tag{6.3}$$

then the scheme is simplified if we take

$$u_3(x,t) = W^1_{(4.9\mathrm{b})}(x,t) + 2^{-1}\left[\frac{\partial}{\partial x}\,\varphi(0,\,0)\right]\big(x - \gamma(t)\big)\,\exp\big(-\alpha\, t\big), \tag{6.4}$$

$$(x,t) \in \overline{G}, \quad \alpha = \alpha_{(4.5)}, \quad \gamma = \gamma_{(4.5)};$$

$u_2^2(x,t) = 0$ for $x < 0$, $t = 0$. In this case, the component $u_1(x,t)$ that contains the boundary layer is absent in the representation (6.1a). Then the solution of problem (2.2), (2.1) takes the form

$$u(x,t) = u_2(x,t) = u_2^1(x,t) + u^2_{2(6.4)}(x,t), \quad (x,t) \in \overline{G}, \tag{6.5a}$$

where $u_2^1(x,t)$ is the solution of the problem

$$\begin{aligned} L\, u_2^1(x,t) &= 0, & (x,t) &\in G \\ u_2^1(x,t) &= \varphi_2(x,t), & (x,t) &\in S; \end{aligned} \tag{6.5b}$$

Here $\varphi_2(x,t) = \varphi(x,t) - u^2_{2(6.4)}(x,t)$, $(x,t) \in S$.

To solve problem (6.5), we use the difference scheme

$$\begin{aligned} \Lambda_{(5.2)}\, z_2^1(x,t) &= 0, & (x,t) &\in G_h, \\ z_2^1(x,t) &= \varphi_2(x,t), & (x,t) &\in S_h, \end{aligned} \tag{6.6a}$$

where $\overline{G}_h$ is the uniform mesh (5.3).

Further, we construct the following special interpolants (similar to (6.2c))

$$\begin{aligned} & u_0^h(x,t) = \overline{z}_2^1(x,t) + u^2_{2(6.4)}(x,t), && (x,t) \in \overline{G}, \\ & p_0^h(x,t) = \overline{p}^{\,h}_{z_2^1}(x,t) + \frac{\partial}{\partial x} u^2_{2(6.4)}(x,t), && (x,t) \in \overline{G}^*, \\ & P_0^h(x,t) = \varepsilon\, p_0^h(x,t), && (x,t) \in \overline{G}^*, \end{aligned} \tag{6.6b}$$

where $\overline{z}_2^1(x,t)$ is a bilinear interpolant constructed using the function $z_2^1(x,t) = z^1_{2(6.6a)}(x,t)$.

The scheme (6.6), (5.3) is the scheme of the singularity splitting method under condition (4.12) or (6.3).

Note that condition (6.3) is satisfied in the case of problem (1.2), (1.3).

6.3. Let us give estimates for the schemes constructed in this section.

In the case of scheme (6.2), (5.3), (5.8), we have the estimates

$$\left| u(x,t) - u_0^h(x,t) \right| \le M \left[N^{-1} \ln N + N_0^{-1+\nu_0} \right], \quad (x,\, t) \in \overline{G}; \tag{6.7a}$$

$$\left| P(x,t) - P_0^h(x,t) \right| \le M \left[N^{-1/2} + N_0^{-1/2} \right], \qquad (x,\, t) \in \overline{G}^*; \tag{6.7b}$$

$$\left| p(x,t) - p_0^h(x,t) \right| \le M \left[N^{-1/2} + N_0^{-1/2} \right], \qquad (x,\, t) \in \overline{G}_0^*, \tag{6.7c}$$

where $\overline{G}_0^* = \overline{G}^*_{0(2.5)}(m)$, m is an arbitrary sufficiently small constant, $M_{(6.7c)} = M(m)$ and $\nu_0 = \nu_{0(5.7)}$.

In the case of scheme (6.6), (5.3) under condition (4.12) or (6.3), the following estimates are valid:

$$\begin{aligned} \left| u(x,t) - u_0^h(x,t) \right| &\le M \left[N^{-1} + N_0^{-1+\nu_0} \right], \quad (x,\, t) \in \overline{G}, \\ \left| p(x,t) - p_0^h(x,t) \right| &\le M \left[N^{-1/2} + N_0^{-1/2} \right], \quad (x,\, t) \in \overline{G}^*, \end{aligned} \tag{6.8}$$

where $\nu_0 = \nu_{0(5.7)}$.

Thus, scheme (6.2), (5.3), (5.8) converges ε-uniformly in $C^1(\overline{G}_0^*)$, and scheme (6.6), (5.3) under condition (4.12) or (6.3) converges ε-uniformly in $C^1(\overline{G}^*)$.

In the case of difference scheme (6.2), (5.3), (5.8), the component $u_{2(6.1a)}(x,t)$ that contains the interior layer and its derivative in x satisfy the estimates

$$\begin{aligned} &\left| u_2(x,t) - u_2^h(x,t) \right| \le M \left[N^{-1} + N_0^{-1+\nu_0} \right], \quad (x,\,t) \in \overline{G}, && (6.9)\\ &\left| p_2(x,t) - p_2^h(x,t) \right| \le M \left[N^{-1/2} + N_0^{-1/2} \right], \quad (x,\,t) \in \overline{G}^{\,*}, \end{aligned}$$

where $p_2(x,t) = \dfrac{\partial}{\partial x} u_2(x,t)$, $p_2^h(x,t) = p^h_{2(6.2\mathrm{d})}(x,t)$. Thus, the component containing the interior layer converges ε-uniformly in $C^1(\overline{G}^{\,*})$.

Theorem 6.1 *Let the assumptions of Theorem* 5.1 *be fulfilled for* $K = 6$. *Then the difference scheme* (6.2), (5.3), (5.8) $\big($*the difference scheme* (6.6), (5.3) *under condition* (4.12) *or* (6.3)$\big)$ *approximates the solution of the problem* (2.2), (2.1), *the derivative* $p(x,t)$ *and the diffusion flux* $P(x,t)$ $\big($*the solution of the problem* (2.2), (2.1) *and the derivative* $p(x,t)\big)$ ε*-uniformly with the estimates* (6.7) *and* (6.9), *respectively* $\big($*with the estimates* (6.8)$\big)$.

Note that the order of the ε-uniform convergence of schemes (6.2), (5.3), (5.8), and (6.6), (5.3) under condition (4.12) or (6.3) is essentially better than it is for the scheme (5.2), (5.8) $\big($see the estimates (5.9), (6.7), (6.8)$\big)$.

In the case of problem (1.1), i.e., the Cauchy problem for the Black-Scholes equation, the scheme of the singularity splitting method makes it possible to obtain the approximation of the solution $C(S,t')$ in a finite neighbourhood of the point (E,T) containing the interior layer, and also of its derivative $(\partial/\partial\, S)\, C(S,t')$ in this neighbourhood excluding the point (E,T), with errors independent of the dimensionless value $\sigma^2 r^{-1}$ for $\sigma^2 r^{-1} \le M$. The interpolants approximating the solution $C(S,t')$ and its derivative $(\partial/\partial\, S)\, C(S,t')$ converge in the maximum norm uniformly with respect to the value $\sigma^2 r^{-1}$ at a rate of convergence with the order close to 1 and 0.5, respectively.

7 Numerical experiments

7.1. In this section, we present experimental results for the problem

$$\begin{aligned} L_{(7.1)}\, u(x,t) \;&\equiv\; \left\{ \varepsilon \frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t} \right\} u(x,t) = 0, \quad (x,t) \in G, && (7.1\text{a})\\ u(x,t) \;&=\; \varphi(x,t), \quad (x,t) \in S, \end{aligned}$$

that has the same singularity of the solution as problem (1.4), (1.3) in a neighbourhood of the interior layer. Note that the problem (1.4), (1.3) is equivalent to problem (1.1). Here

$$G = D \times (0,\, T], \quad D = \{x :\; x \in (-d,\, d)\}, \tag{7.1b}$$

$$T = 1, \quad d = 2; \quad \varphi(x,0) = \begin{cases} 0, & -d < x \le 0, \\ x + m\,x^2, & 0 < x < d; \end{cases}$$

$$\begin{aligned} &\varphi(-d,t) = 0,\\ &\varphi(d,t) = e^{-t}\Big(md^2 + \big(2mt(1-\varepsilon) + 1\big)d + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2 t^2 \Big), \end{aligned}$$

where $m = 4^{-1}$.

The jump of the first order derivative of the function $\varphi_{(7.1)}(x,t)$ at the point $(0,t)$ is the same as for the function $\varphi_{(1.3a)}(x)$ for $x = 0$ and is equal to 1 for both functions.

Note that $\varphi(x,t) = w(x,t)$ for $(x,t) \in S$, $x \geq 0$, where

$$w(x,t) = e^{-t}\Big(mx^2 + \big(2mt(1-\varepsilon)+1\big)x + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2t^2\Big),$$

$$(x,t) \in \mathbb{R} \times [0,\, T].$$

Here the function $w(x,t)$ is the solution of the Cauchy problem

$$\begin{aligned} L_{(7.1)}\, w(x,t) &= 0, \qquad (x,t) \in \mathbb{R} \times (0,\, T], \\ w(x,0) &= \varphi_w(x), \quad x \in \mathbb{R}, \end{aligned} \tag{7.2}$$

where $\varphi_w(x) = x + m\,x^2$, $x \in \mathbb{R}$.

The choice of boundary conditions for $x = -d$, d ensures that compatibility conditions are fulfilled for the data of problem (7.1) and prevents the appearance of the boundary layer and of the interior layer in a neighbourhood of the characteristic passing through the point $(d, 0)$.

Under the chosen data of the problem, the singularity of the solution generated by the jump of the derivative of the initial function is not "polluted" by other singularities, that allows us to study numerically the efficiency of the constructed difference scheme in a domain containing the interior layer and to compare this scheme with the classical finite difference scheme.

The data of problem (7.1) satisfy condition (6.3) $\big($and condition (4.12)$\big)$. Thus, for the numerical solution of this problem, it is possible to apply the simplified scheme (6.6), (5.3), i.e., the scheme of the singularity splitting method under condition (6.3) or (4.12) $\big($we denote it briefly by Scheme A$\big)$.

In order to estimate the efficiency of the developed method, we compare solutions generated using Scheme A in accuracy with discrete solutions of problem (7.1) generated using the classical finite difference scheme (5.2), (5.3) $\big($we denote it briefly by Scheme B$\big)$.

The plots of the solutions a_1, b_1 and the derivatives a_2, b_2 computed using Scheme A $\big($see a_1, $a_2\big)$ and Scheme B $\big($see b_1, $b_2\big)$ are presented on Fig. 1 for $\varepsilon = 2^{-10}$, $N = 16$ and $N_0 = 16$.

7.2. To analyze errors in the discrete solutions, a technique similar to that given in [1] is used, however, it is modified with regard to the singularity splitting method. Computations are made for values of $\varepsilon = 2^{-j}$, $j = 0, 1, \ldots, 34$ on grids with the number of nodes $N = N_0$ for $N = 2^i$, $i = 5, 6, \ldots, 10$. The numerical solution $u^{h,\,N^F}_{0,\,\varepsilon}(x,t)$, generated by Scheme A on the finest mesh $\overline{G}^{N^F}_h$ with $N = N_0 = N^F = 2048$ for each value of ε is used as the exact solution of problem (7.1).

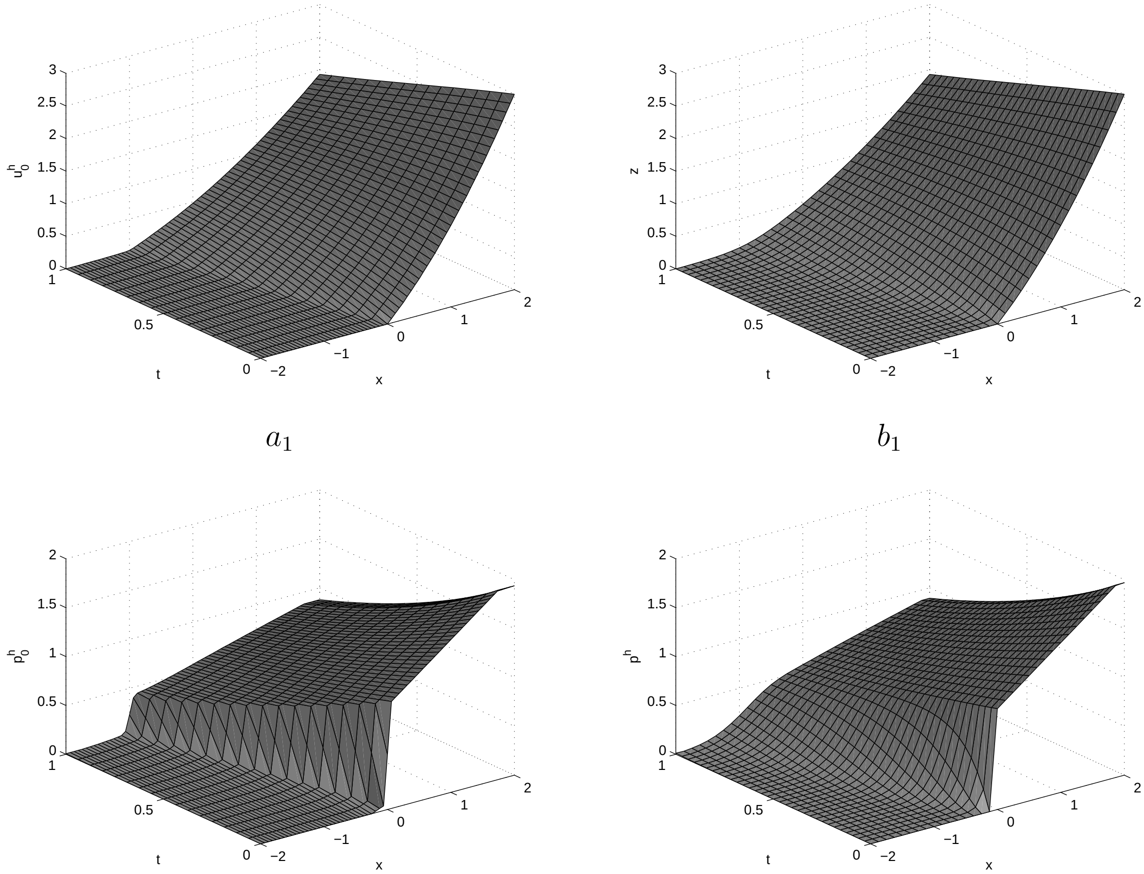
Errors in the numerical solutions in the maximum norm for each value of ε and N are computed by the formula

$$E^N_\varepsilon = E^N_\varepsilon\big(u^{h,\,N}_{0,\,\varepsilon}(\cdot)\big) = \big\|u^{h,\,N^F}_{0,\,\varepsilon}(x,t) - u^{h,\,N}_{0,\,\varepsilon}(x,t)\big\|_{\,\overline{G}^N_h} \tag{7.3}$$

for Scheme A and by the formula

$$E^N_\varepsilon = E^N_\varepsilon\big(z^N_\varepsilon(\cdot)\big) = \big\|u^{h,\,N^F}_{0,\,\varepsilon}(x,t) - z^N_\varepsilon(x,t)\big\|_{\,\overline{G}^N_h} \tag{7.4}$$

for Scheme B. Here the function $u^{h,\,N}_{0,\,\varepsilon}(x,t) = u^{h,\,N}_{0(6.6),\,\varepsilon}(x,t)$ in (7.3) and the function $z^N_\varepsilon(x,t)$ in (7.4) are the numerical solutions obtained, respectively, by Schemes A and B.



a_1 b_1

a_2 b_2

Figure 1: Plots of the solutions a_1, b_1 and the derivatives a_2, b_2; plots of a_1, a_2 and b_1, b_2 are generated by Schemes A and B, respectively, for $\varepsilon = 2^{-10}$, $N = 16$ and $N_0 = 16$.

Tables 1 and 2 contain the values E_ε^N of errors in the solutions generated by Schemes A and B for various values of ε and N. The value E^N in the last rows of the tables is the maximal value of the errors E_ε^N with respect to ε, corresponding to the given value of N.

Tables 3 and 4, which are similar to tables 1 and 2, demonstrate errors in the first derivatives computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p_{0,\,\varepsilon}^{h,\,N}(\cdot)\right) = \left\|p_{0,\,\varepsilon}^{h,\,N^F}(x,t) - p_{0,\,\varepsilon}^{h,\,N}(x,t)\right\|_{\,\overline{G}_h^{N\,*}}, \tag{7.5}$$

$$\overline{G}_h^{N\,*} = \overline{G}_h^{N} \setminus S^{(*)}, \quad S^{(*)} = S^{(*)}_{(2.2c)}$$

for Scheme A and by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p_{z,\,\varepsilon}^{N}(\cdot)\right) = \left\|p_{0,\,\varepsilon}^{h,\,N^F}(x,t) - p_{z,\,\varepsilon}^{N}(x,t)\right\|_{\,\overline{G}_h^{N\,\{*\}}}, \tag{7.6}$$

$$\overline{G}_h^{N\,\{*\}} = \overline{G}_h^{N} \setminus S^{\{*\}}, \quad S^{\{*\}} = \{(x,0):\ x = x_{i-1},\, x_i,\, x_{i+1};\ x_i = 0\}$$

for Scheme B. Here, $p_{z,\,\varepsilon}^{N}(x,t)$ in (7.6) is the first difference derivative

$$p_{z,\,\varepsilon}^{N}(x_i,\, t_j) = \frac{z_\varepsilon^N(x_{i+1},\, t_j) - z_\varepsilon^N(x_i,\, t_j)}{x_{i+1} - x_i}, \quad i = 0, \ldots, N, \quad j = 0, \ldots, N_0. \tag{7.7}$$

Table 1: Errors $E_\varepsilon^N = E_\varepsilon^N(u_{0,\varepsilon}^{h,N})$ and $E^N = E^N(u_{0,\varepsilon}^{h,N})$ for the solutions generated by Scheme A.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1320-02	0.6633-03	0.3328-03	0.1666-03	0.8338-04	0.4170-04
2^{-1}	0.1140-02	0.5863-03	0.2972-03	0.1496-03	0.7507-04	0.3760-04
2^{-2}	0.2782-02	0.1426-02	0.7219-03	0.3632-03	0.1822-03	0.9123-04
2^{-3}	0.3904-02	0.2006-02	0.1017-02	0.5120-03	0.2569-03	0.1287-03
2^{-4}	0.4614-02	0.2384-02	0.1214-02	0.6131-03	0.3082-03	0.1545-03
2^{-5}	0.5034-02	0.2623-02	0.1345-02	0.6823-03	0.3440-03	0.1727-03
2^{-6}	0.5288-02	0.2769-02	0.1429-02	0.7289-03	0.3688-03	0.1856-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-15}	0.5557-02	0.2948-02	0.1544-02	0.8004-03	0.4116-03	0.2103-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-34}	0.5558-02	0.2948-02	0.1544-02	0.8006-03	0.4117-03	0.2104-03
E^N	**0.5558-02**	**0.2948-02**	**0.1544-02**	**0.8006-03**	**0.4117-03**	**0.2104-03**

Table 2: Errors $E_\varepsilon^N = E_\varepsilon^N(z_\varepsilon^N)$ and $E^N = E^N(z_\varepsilon^N)$ for the solutions generated by Scheme B.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.9868-02	0.5482-02	0.3296-02	0.2116-02	0.1419-02	0.9756-03
2^{-1}	0.8081-02	0.4425-02	0.2560-02	0.1584-02	0.1036-02	0.7016-03
2^{-2}	0.6600-02	0.3718-02	0.2093-02	0.1237-02	0.7772-03	0.5126-03
2^{-3}	0.9033-02	0.4972-02	0.2627-02	0.1354-02	0.6878-03	0.3850-03
2^{-4}	0.1242-01	0.7186-02	0.3933-02	0.2072-02	0.1065-02	0.5406-03
2^{-5}	0.1515-01	0.9293-02	0.5348-02	0.2919-02	0.1535-02	0.7887-03
2^{-6}	0.1706-01	0.1105-01	0.6736-02	0.3866-02	0.2107-02	0.1107-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-15}	0.1963-01	0.1403-01	0.9948-02	0.7029-02	0.4956-02	0.3486-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-34}	0.1964-01	0.1404-01	0.9960-02	0.7045-02	0.4978-02	0.3516-02
E^N	**0.1964-01**	**0.1404-01**	**0.9960-02**	**0.7045-02**	**0.4978-02**	**0.3516-02**

The function $p_{0,\varepsilon}^{h,N^F}(x,t)$ in formulae (7.5) and (7.6) and the function $p_{0,\varepsilon}^{h,N}(x,t)$ in formula (7.5) are the special interpolants of the first order derivative of the solution computed by formula (6.6), respectively, on the finest mesh $\overline{G}_h^{N^F}$ and on the mesh $\overline{G}_h^N$ for fixed value of ε.

Analyzing the values of errors for the solutions in Tables 1 and 2, and for the first derivatives in Table 3, we observe the ε-uniform convergence, since, with decreasing ε, the errors are stabilized for each value of N approximately for one and the same values of ε, i.e., the errors are independent of the value of the parameter ε, moreover, the values of E^N (the last row) decrease as N increases. However, in Table 4 the first derivative of the solution generated by Scheme B does not converge at all; the values of E^N practically do not change as N increases.

In Tables 5 and 6, the values of q_ε^N are shown that are the convergence orders for the solutions computed by Schemes A and B, respectively. In analogous Table 7, one can see the convergence orders for the first discrete derivatives generated by Scheme A for various values of ε and N. The value q^N in last rows of the tables is the minimal value of q_ε^N with respect to ε, corresponding to the given value of N.

The convergence order for the discrete solutions is defined by the formula

$$q_\varepsilon^N = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}. \tag{7.8}$$

The quantities E_ε^N, E_ε^{2N} are defined by formula (7.3) for Scheme A and by formula (7.4) for Scheme B.

The convergence order for the discrete derivatives is defined by formula (7.8) where the quantities E_ε^N, E_ε^{2N} are defined by formula (7.5) for Scheme A and by formula (7.6) for Scheme B.

Table 3: Errors $E_\varepsilon^N = E_\varepsilon^N(p_{0,\,\varepsilon}^{h,\,N})$ and $E^N = E^N(p_{0,\,\varepsilon}^{h,\,N})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1562-01	0.7931-02	0.4073-02	0.2106-02	0.1100-02	0.5835-03
2^{-1}	0.1562-01	0.7910-02	0.4016-02	0.2034-02	0.1048-02	0.5470-03
2^{-2}	0.1620-01	0.8583-02	0.4431-02	0.2253-02	0.1136-02	0.5705-03
2^{-3}	0.1701-01	0.9262-02	0.4858-02	0.2492-02	0.1263-02	0.6355-03
2^{-4}	0.1754-01	0.9768-02	0.5198-02	0.2690-02	0.1370-02	0.6915-03
2^{-5}	0.1788-01	0.1011-01	0.5446-02	0.2840-02	0.1454-02	0.7359-03
2^{-6}	0.1806-01	0.1034-01	0.5616-02	0.2947-02	0.1515-02	0.7688-03
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-15}	0.1831-01	0.1066-01	0.5915-02	0.3966-02	0.2697-02	0.1850-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	0.1831-01	0.1066-01	0.5920-02	0.3974-02	0.2708-02	0.1865-02
E^N	**0.1831-01**	**0.1066-01**	**0.5920-02**	**0.3974-02**	**0.2708-02**	**0.1865-02**

Table 4: Errors $E_\varepsilon^N = E_\varepsilon^N(p_{z,\,\varepsilon}^N)$ and $E^N = E^N(p_{z,\,\varepsilon}^N)$ for the first discrete derivatives generated by Scheme B.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1080+00	0.7864-01	0.6165-01	0.5118-01	0.4465-01	0.4000-01
2^{-1}	0.1296+00	0.1013+00	0.7555-01	0.5941-01	0.5017-01	0.4405-01
2^{-2}	0.1394+00	0.1219+00	0.9764-01	0.7388-01	0.5826-01	0.4964-01
2^{-3}	0.1302+00	0.1315+00	0.1178+00	0.9572-01	0.7301-01	0.5768-01
2^{-4}	0.1045+00	0.1228+00	0.1275+00	0.1157+00	0.9473-01	0.7256-01
2^{-5}	0.7901-01	0.9761-01	0.1189+00	0.1254+00	0.1147+00	0.9424-01
2^{-6}	0.8714-01	0.7835-01	0.9407-01	0.1170+00	0.1244+00	0.1141+00
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-15}	0.9914-01	0.9833-01	0.9985-01	0.1005+00	0.1006+00	0.1002+00
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	0.9918-01	0.9839-01	0.1000+00	0.1008+00	0.1012+00	0.1014+00
E^N	**0.1394+00**	**0.1315+00**	**0.1275+00**	**0.1254+00**	**0.1244+00**	**0.1238+00**

Orders of the rate of ε-uniform convergence for the solutions generated by Schemes A and B (see Tables 5 and 6) are close, respectively, to 1 and 0.5; for the first derivative generated by Scheme A (see Table 7), the order of the rate of ε-uniform convergence is close to 0.5.

Table 5: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(u_{0,\,\varepsilon}^{h,\,N})$ and $q^N = q^N(u_{0,\,\varepsilon}^{h,\,N})$ for the solutions of Scheme A.

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.9928	0.9950	0.9983	0.9986	0.9997
2^{-1}	0.9593	0.9802	0.9903	0.9948	0.9975
2^{-2}	0.9641	0.9821	0.9910	0.9952	0.9979
2^{-3}	0.9606	0.9800	0.9901	0.9949	0.9972
2^{-4}	0.9526	0.9736	0.9856	0.9923	0.9963
2^{-5}	0.9405	0.9636	0.9791	0.9880	0.9941
2^{-6}	0.9334	0.9544	0.9712	0.9829	0.9906
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-15}	0.9146	0.9331	0.9479	0.9595	0.9688
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-34}	0.9148	0.9331	0.9475	0.9595	0.9685
q^N	**0.9148**	**0.9331**	**0.9475**	**0.9595**	**0.9685**

Table 6: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(z_\varepsilon^N)$ and $q^N = q^N(z_\varepsilon^N)$ for the solutions of Scheme B.

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.8481	0.7340	0.6394	0.5765	0.5405
2^{-1}	0.8689	0.7895	0.6926	0.6125	0.5623
2^{-2}	0.8279	0.8290	0.7587	0.6705	0.6005
2^{-3}	0.8614	0.9204	0.9562	0.9772	0.8371
2^{-4}	0.7894	0.8696	0.9246	0.9602	0.9782
2^{-5}	0.7051	0.7971	0.8735	0.9272	0.9607
2^{-6}	0.6266	0.7141	0.8011	0.8757	0.9285
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-15}	0.4845	0.4960	0.5011	0.5041	0.5076
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-34}	0.4843	0.4953	0.4995	0.5010	0.5016
q^N	**0.4843**	**0.4953**	**0.4995**	**0.5010**	**0.5016**

7.3. Thus, it follows from our numerical experiments that the solution and its first order derivative obtained by Scheme A, i.e., the scheme based on the method of the additive splitting of a singularity, converge ε-uniformly at the rate of convergence of the order close to 1 and 0.5, respectively. Whereas the convergence rate for the solutions of the classical finite difference Scheme B yields to that for scheme A, and the derivatives computed by Scheme B do not converge even for fixed values of the parameter ε. The numerical experiments, consistent with the theoretical results, illustrate the efficiency of the singularity splitting method for the approximation of the interior layer generated by the discontinuity of the first order derivative $\dfrac{\partial}{\partial x}\varphi(x,0)$ in problem (7.1).

The numerical experiments showed that the special difference scheme constructed in this paper, i.e., the scheme of the method of the additive splitting of a singularity on uniform meshes, is effective both for small values of N and for its sufficiently large values, for which results of the theoretical study become apparent.

Table 7: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(p_{0,\varepsilon}^{h,N})$ and $q^N = q^N(p_{0,\varepsilon}^{h,N})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.9778	0.9614	0.9516	0.9370	0.9147
2^{-1}	0.9816	0.9779	0.9814	0.9567	0.9380
2^{-2}	0.9164	0.9538	0.9758	0.9879	0.9937
2^{-3}	0.8770	0.9310	0.9631	0.9804	0.9909
2^{-4}	0.8445	0.9101	0.9504	0.9734	0.9864
2^{-5}	0.8226	0.8925	0.9393	0.9659	0.9824
2^{-6}	0.8046	0.8806	0.9303	0.9599	0.9786
⋮	⋮	⋮	⋮	⋮	⋮
2^{-15}	0.7804	0.8498	0.5767	0.5563	0.5438
⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	0.7804	0.8485	0.5750	0.5534	0.5381
q^N	**0.7804**	**0.8485**	**0.5750**	**0.5534**	**0.5381**

In the case of the Cauchy problem for the Black-Scholes equation, the interpolants constructed using solutions of the special difference scheme, which approximates the solution $C(S,t')$ of problem (1.1) and its derivative $(\partial/\partial\, S)\, C(S,t')$ (for $(S,t') \neq (E,T)$) in a neighbourhood of the interior layer, converge at the rate of $\sigma^2\, r^{-1}-$uniform convergence with orders close to 1 and 0.5, respectively.

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