

# Topological Aspects of Poset Spaces

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## Abstract

We study two classes of spaces whose points are filters on partially ordered sets. Points in MF spaces are maximal filters, while points in UF spaces are unbounded filters. We give a thorough account of the topological properties of these spaces. We obtain a complete characterization of the class of countably based MF spaces: they are precisely the second-countable  $T_1$  spaces with the strong Choquet property. We apply this characterization to domain theory to characterize the class of second-countable spaces with a domain representation.

## 1 Introduction

Recent work in mathematical logic [8, 9, 10] has led to an interest in certain topological spaces formed from filters on partially ordered sets. This paper describes the general topology of these poset spaces.

The results of the paper are divided as follows. In Section 2, we define two classes of spaces, MF spaces and UF spaces, which together are known as poset spaces. We show that many familiar spaces are homeomorphic to poset spaces. In Section 3, we characterize the separation properties of poset spaces and show that any second-countable poset space is homeomorphic to a space of the same kind formed from a countable poset. In Section 4, we

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show that MF spaces are closed under arbitrary topological products and  $G_\delta$  subspaces. We show that UF spaces are not closed under binary products. In Section 5, we establish two completeness properties (the property of Baire and the strong Choquet property) that all poset spaces possess. We give a characterization of the class of countably based MF spaces as the class of second-countable  $T_1$  spaces with the strong Choquet property. In Section 6, we apply the results of Section 5 to domain theory, giving a complete characterization of the second-countable topological spaces that have a domain representation. Section 7 contains results on the relationship between MF spaces (not necessarily countably based) and semi-topogenous orders. We use semi-topogenous orders to establish a sufficient condition for an arbitrary space to be homeomorphic to an MF space. In Section 8, we show that every second-countable poset space is either countable or contains a perfect closed set.

## 2 Poset spaces

Our goal in this section is to define the class of poset spaces and show that this class includes all complete metric spaces and all locally compact Hausdorff spaces. We first review some basic definitions about partially ordered sets.

A *poset* is a set  $P$  with an antireflexive, transitive relation  $\preceq$  such that the following conditions hold for all  $p, q$  and  $r$  in  $P$ .

- (1)  $p \preceq p$ .
- (2) If  $p \preceq q$  and  $q \preceq p$  then  $q = p$ .
- (3) If  $p \preceq q$  and  $q \preceq r$  then  $p \preceq r$ .

We write  $p \prec q$  if  $p \preceq q$  and  $p \neq q$ . If there is no  $r$  such that  $r \preceq p$  and  $r \preceq q$  then we write  $p \perp q$ .

A *filter* is a subset  $F$  of a poset  $P$  satisfying the following two conditions.

- (1) For every  $p, q \in F$  there is an  $r \in F$  such that  $r \preceq p$  and  $r \preceq q$ .
- (2) For every  $p \in F$  and  $q \in P$  if  $p \prec q$  then  $q \in F$ .

A filter  $F$  is *unbounded* if there is no  $r \in P$  such that  $r \prec q$  for every  $q \in F$ . Furthermore,  $F$  is *maximal* if there is no strictly larger filter containing  $F$ .

Every maximal filter is unbounded, but in general not every unbounded filter is maximal.

For any poset  $P$ , we let  $\text{UF}(P)$  denote the set of unbounded filters on  $P$  and let  $\text{MF}(P)$  denote the set of maximal filters on  $P$ . We topologize  $\text{UF}(P)$  with the basis  $\{N_p \mid p \in P\}$ , where

$$N_p = \{F \in \text{UF}(P) \mid p \in F\}.$$

We give  $\text{MF}(P)$  the topology it inherits as a subset of  $\text{UF}(P)$ ; when we work with spaces of maximal filters we may write  $N_p$  to denote the set of maximal filters containing  $p$ . To facilitate the exposition, we sometimes identify  $p \in P$  with the open set  $N_p$  and identify a subset  $U$  of  $P$  with the open set  $\bigcup_{p \in U} N_p$ .

A *UF space* is a space of the form  $\text{UF}(P)$  and an *MF space* is a space of the form  $\text{MF}(P)$ . UF spaces and MF spaces are collectively referred to as *poset spaces*. A poset space is *countably based* if it is formed from a countable poset. It is possible that  $P$  is uncountable but  $\text{MF}(P)$  or  $\text{UF}(P)$  is a second-countable space. We will show below that every second-countable poset space is homeomorphic to a countably based poset space. This result justifies our terminology.

We now present two examples showing that many familiar spaces are homeomorphic to poset spaces.

**Theorem 2.1.** Every locally compact Hausdorff space is homeomorphic to an MF space.

**Proof.** Let  $X$  be a locally compact Hausdorff space and let  $P$  be the set of all precompact open subsets of  $X$ . For  $U, V \in P$  we put  $U \preceq V$  if the closure of  $U$  is contained in  $V$ . It is not difficult to prove that if  $F$  is any maximal filter on  $P$  then  $\bigcap F$  contains exactly one point and that the map  $F \mapsto \bigcap F$  is a homeomorphism from  $\text{MF}(P)$  to  $X$ .  $\square$

As there are non-locally-compact complete separable metric spaces and locally compact Hausdorff nonmetrisable spaces, the next theorem is independent of Theorem 2.1. We remark that a construction similar to that in the next theorem was used by Lawson [6] to represent complete separable metric spaces in the context of domain theory (see Section 6).

**Theorem 2.2.** For every complete metric space  $X$  there is a poset  $P$  such that  $X \cong \text{UF}(P)$  and  $\text{UF}(P) = \text{MF}(P)$ . Moreover, if  $X$  is infinite then we may take the cardinality of  $P$  to be that of any dense subset of  $X$ .

**Proof.** Let  $X$  be a complete metric space; we write  $B(x, \epsilon)$  for the open metric ball of radius  $\epsilon > 0$  around a point  $x \in X$ . Let  $A$  be a dense subset of  $X$ . The poset  $P$  is the set of all open balls  $B(a, r)$  where  $r$  is a positive rational number and  $a \in A$ . For  $p = B(a, r)$  and  $p' = B(a', r')$  in  $P$  we let  $p \prec p'$  if and only if  $d(a, a') + r < r'$ . It is not difficult to prove that if  $F$  is any unbounded filter on  $P$  then  $\bigcap F$  contains exactly one point and that the map sending  $F$  to  $\bigcap F$  is a homeomorphism from  $\text{UF}(P)$  to  $X$ . It is also straightforward to prove the every unbounded filter on  $P$  is maximal. Clearly the cardinality of  $P$  is that of  $A$  if  $A$  is infinite. Furthermore,  $A$  is infinite if  $X$  is infinite.  $\square$

There are second-countable nonmetrisable Hausdorff MF spaces. One example is the Gandy–Harrington space from modern descriptive set theory (see [9]).

### 3 Separation and countability properties

In this section, we determine the separation properties that a poset space must satisfy. We then show that every second countable poset space is homeomorphic to a poset space obtained from a countable poset. In Section 8, we will show that a countably based poset space is either countable or contains a perfect closed set.

- Theorem 3.1.** (1) Every UF space is  $T_0$ .
- (2) Every MF space is  $T_1$ .
- (3) If  $\text{UF}(P)$  is  $T_1$  then every unbounded filter on  $P$  is maximal and thus  $\text{UF}(P) = \text{MF}(P)$ .

**Proof.** (1) follows from the fact that distinct filters are distinct as subsets of  $P$ . (2) follows from the fact that no maximal filter can properly contain another maximal filter. To prove (3), suppose  $\text{UF}(P)$  is  $T_1$  and let  $F$  be an unbounded filter on  $P$ . Let  $G$  be a filter on  $P$  such that  $F \subseteq G$ . Clearly  $G$  is unbounded. If  $F \neq G$  then there must be a  $p \in P$  such that  $F \in N_p$  and  $G \notin N_p$ . This means  $p \in (F \setminus G)$ , which is impossible. Thus  $F = G$ ; this shows that  $F$  is maximal.  $\square$

**Theorem 3.2.** Suppose that  $P$  is a poset such that  $\text{MF}(P)$  is second countable. There is a countable subposet  $R$  of  $P$  such that the map  $F \mapsto R \cap F$  is a homeomorphism from  $\text{MF}(P)$  to  $\text{MF}(R)$ .

**Proof.** Suppose that  $\text{MF}(P)$  is second countable; thus  $P$  contains a countable subset  $Q_0$  such that  $\{N_q : q \in Q_0\}$  is a basis for the topology, because every basis of a second-countable topology contains a countable subclass which is also a basis. For  $n = 0, 1, 2, \dots$ , we construct a set  $Q_{n+1}$  inductively to satisfy the following conditions.

- $Q_{n+1}$  is countable.
- $Q_n \subseteq Q_{n+1} \subseteq P$ .
- For every  $F \in \text{MF}(P)$  and every finite subset  $D \subseteq Q_n \cap F$  there is a  $q \in Q_{n+1}$  such that  $q \preceq d$  for all  $d \in D$ .

In order to see that  $Q_{n+1}$  can be taken to be countable, suppose  $D$  is a finite subset of  $Q_n$  with nonempty intersection. Let  $E_D$  be the set of all  $p \in P$  such that  $p \preceq d$  for every  $d \in D$ . For every filter  $F \in \text{MF}(P)$  with  $D \subseteq F$  there is an element  $p \in E_D \cap F$ ; thus  $\{N_e : e \in E_D\}$  is an open cover of the intersection of all open sets  $N_d$  with  $d \in D$ . Since the given space is second countable, there is a countable subset  $F_D$  of  $E_D$  covering the same set of maximal filters; if some finite subset  $D$  of  $Q_n$  is not contained in any filter then let  $F_D$  be empty. Now take  $Q_{n+1}$  to be the union of all  $F_D$  where  $D \subseteq Q_n$  and  $D$  is finite;  $Q_{n+1}$  is also at most countable.

Let  $R = \bigcup_i Q_i$ . Note that  $R$  is countable and  $\{N_r \mid r \in R\}$  is a basis for  $\text{MF}(P)$ . For  $F \subseteq P$  we write  $\phi(F)$  for  $F \cap R$ . It is straightforward to verify that  $\phi(F)$  is a filter for every  $F \in \text{MF}(P)$ , by the construction of  $R$ . Because  $R \subseteq P$ , every  $F \in \text{MF}(R)$  extends to some  $F' \in \text{MF}(P)$ ; then  $\phi(F') = F$ . This shows that  $\phi$  determines a surjective map  $\Phi$  from  $\text{MF}(P)$  to  $\text{MF}(R)$ .

In order to prove that  $\Phi$  is injective, it suffices to prove the following statement. For all filters  $V, W$  on  $P$  we have  $V \subseteq W$  if and only if  $\phi(V) \subseteq \phi(W)$ . Suppose  $p \in V \setminus W$ . Then  $W \not\subseteq N_p$  and thus  $W \not\subseteq N_q$  for all  $q$  with  $N_q \subseteq N_p$ . On the other hand,  $R$  is a basis and  $N_p$  is the union of basic open sets. Since  $V \in N_p$  there is a  $r \in R$  with  $N_r \subseteq N_p$  and  $V \in N_r$ . It follows that  $r \in \phi(V) \setminus \phi(W)$ . The other direction of the implication is trivial.

This shows that  $\Phi$  is a bijection from  $\text{MF}(P)$  to  $\text{MF}(R)$ . It is straightforward to show that  $\Phi$  is a homeomorphism.  $\square$

**Corollary 3.3.** An MF space is homeomorphic to a countably based MF space if and only if it is second countable.

**Corollary 3.4.** A UF space is homeomorphic to a countably based UF space if and only if it is second countable.

**Proof.** Let  $X = \text{UF}(P)$  be second countable. Construct a poset  $R$  and a map  $\phi$  in a manner analogous to the proof of Theorem 3.2. We show that  $\phi$  is a homeomorphism from  $\text{UF}(P)$  to  $\text{UF}(R)$ . It is clear that if  $F \in \text{UF}(P)$  then  $\phi(F) \in \text{UF}(R)$ . Every  $G \in \text{UF}(R)$  extends to some  $G' \in \text{UF}(P)$  and then  $\phi(G') = G$ . Thus  $\phi$  is well defined and surjective as a map from  $\text{UF}(P)$  to  $\text{UF}(R)$ . To see that  $\phi$  is injective, suppose that  $F \neq G$  are unbounded filters on  $P$ . Without loss of generality we may assume there is some  $p \in G \setminus F$ . There is thus some  $r$  in  $R \cap (G \setminus F)$ , because  $R$  is a basis. But  $r \in R \cap (G \setminus F)$  implies  $r \in \phi(G) \setminus \phi(F)$ , which shows  $\phi(G) \neq \phi(F)$ . Thus  $\phi$  is a bijection from  $\text{UF}(P)$  to  $\text{UF}(R)$ . The proof that  $\phi$  is a homeomorphism is straightforward.  $\square$

## 4 Product and subspace properties

In this section, we show that the class of MF spaces is closed under taking  $G_\delta$  subsets and arbitrary topological products. The class of UF spaces is closed under taking  $G_\delta$  subspaces but it not closed under even finite products.

**Theorem 4.1.** The class of MF spaces is closed under arbitrary topological products.

**Proof.** Suppose that we are given a collection  $\langle \langle P_i, \preceq_i \rangle \mid i \in I \rangle$  of posets. We may assume without loss of generality that each poset has a greatest element, which we denote by  $p_i$ . We form a poset  $P$  consisting of those functions  $f$  from  $I$  to  $\bigcup_{i \in I} P_i$  such that  $f(i) \in P_i$  for all  $i$  and  $f(i) = p_i$  for all but finitely many  $i$ . For  $f, g \in P$  we put  $f \preceq g$  if  $f(i) \preceq_i g(i)$  for all  $i$ .

We define a map from  $\prod_i \text{MF}(P_i)$  to  $\text{MF}(P)$  by sending  $\prod_i F_i$  to the set of all functions  $f \in P$  such that  $f(i) \in F_i$  for all  $i$ . It is straightforward to show that this map is a homeomorphism from  $\prod_i \text{MF}(P_i)$  to  $\text{MF}(P)$ .  $\square$

**Corollary 4.2.** Every topological product of countably many countably based MF spaces is homeomorphic to a countably based MF space.

**Proof.** Under these hypotheses, the poset constructed in Theorem 4.1 is countable.  $\square$

**Theorem 4.3.** The class of MF spaces is closed under taking  $G_\delta$  subspaces.

**Proof.** Suppose that  $\langle U_i \mid i \in \mathbb{N} \rangle$  is a sequence of open subsets of  $\text{MF}(P)$  and  $\bigcap_i U_i$  is nonempty. We form a poset  $Q$  of pairs  $\langle n, p \rangle$  such that  $n \in \mathbb{N}$  and  $N_p \subseteq \bigcap_{i < n} U_i$ . We let  $\langle n, p \rangle \prec \langle n', p' \rangle$  if  $n > n'$  and  $p \preceq p'$ . We define a map from  $\bigcap_i U_i$  to  $\text{MF}(Q)$  by sending a maximal filter  $F$  to the set of all  $\langle n, p \rangle$  in  $Q$  such that  $F \in N_p$ . It is not difficult to show that this map is a homeomorphism from  $\bigcap_i U_i$  to  $\text{MF}(Q)$ .  $\square$

Theorem 4.3 gives an optimal result. We will show below that all poset spaces have the property of Baire. The real line is homeomorphic to a UF space, but the  $F_\sigma$  subset of rational numbers does not have the property of Baire and therefore is not homeomorphic to a poset space.

The class of UF spaces does not enjoy the closure properties that the class of MF spaces does. We now give an example showing that the class of UF spaces is not closed under finite products.

**Theorem 4.4.** The class of UF spaces is not closed under finite products. There are two posets  $P, Q$  such that  $\text{MF}(P) = \text{UF}(P)$ ,  $\text{MF}(Q) = \text{UF}(Q)$ , but the topological product  $\text{MF}(P) \times \text{MF}(Q)$  is not homeomorphic to any UF space.

**Proof.** Let  $\omega$  denote the least infinite countable ordinal and let  $\omega_1$  denote the least uncountable ordinal. We define  $P$  to be the set of functions from finite initial segments of  $\omega$  to  $\{0, 1\}$  and define  $Q$  to be the set of functions from countable initial segments of  $\omega_1$  to  $\{0, 1\}$ . For both posets the relation  $\preceq$  is given by extension:  $p \preceq q$  if for all  $\alpha$  in the domain of  $q$ ,  $p(\alpha)$  is defined and takes the value  $q(\alpha)$ .

We first show that  $\text{MF}(P) = \text{UF}(P)$  and  $\text{MF}(Q) = \text{UF}(Q)$ . Assume that  $F$  is an unbounded filter on  $P$  (the argument for  $Q$  is parallel). Then all functions in  $F$  are compatible, that is, they do not contradict each other on any value in the intersection of their domains. There is thus a total limit function  $f$ , because otherwise there would be a first ordinal  $\alpha$  where  $f$  is undefined and the function extending  $f$  which maps  $\alpha$  to 0 would define an element of  $P$  which would be a lower bound for the filter  $F$ . Since  $f$  is total, all functions mapping the ordinals up to some  $\alpha$  in the domain of  $f$  to the corresponding value of  $f$  are in the filter. One can see that this filter is

already maximal, because any element outside it but still in  $P$  is incompatible with this function and adding it would destroy the filter property.

Assume now, by way of contradiction, that  $\text{UF}(P) \times \text{UF}(Q)$  is homeomorphic to a space  $\text{UF}(R)$ . We denote by  $\pi_P, \pi_Q$  the continuous, open projection maps from  $\text{UF}(R)$  to its factor spaces. There is a filter  $F$  in  $\text{UF}(R)$  such that  $\pi_P(F)$  and  $\pi_Q(F)$  are the filters generated by the set of all functions in  $P$  and  $Q$ , respectively, which map all inputs to 0. Now one can select an infinite sequence  $r_0, r_1, \dots$  in  $F$  such that for each  $n$  the projection  $\pi_P(N_{r_n})$  consists only of functions which map the first  $n$  numbers to 0 and  $r_{n+1} \preceq r_n$  for all  $n$ . The sequence  $\langle r_i \rangle$  generates a subfilter  $G \subseteq F$ . There is no lower bound  $r$  for  $G$ , because otherwise  $\pi_P(N_r)$  would be an open set containing some basic open set  $N_p$  such that  $N_p \subseteq \pi_P(N_{r_n})$  for all  $n$ ; such a  $p$  cannot exist by construction.

On the other hand, there is a function  $f$  contained in all the open sets  $\pi_Q(N_{r_n})$  and there are basic open neighbourhoods of  $f$  generated by  $q_0, q_1, \dots$  such that  $N_{q_n} \subseteq \pi_Q(N_{r_n})$  for each  $n$ . The basic open sets  $N_{q_0}, N_{q_1}, \dots$  fix  $f$  only on countably many ordinals and thus their intersection is also a basic open set. So  $\pi_P(G)$  is bounded while  $\pi_Q(F)$  is not and thus  $G \subset F$ . It follows that  $\text{UF}(R)$  is not a  $T_1$  space. This contradicts the assumption that  $\text{UF}(R)$  is homeomorphic to  $\text{MF}(P) \times \text{MF}(Q)$ .  $\square$

We note that the example in the previous theorem was not second countable and that the failure of second countability was important to the proof.

**Question 4.5.** Is the class of countably based UF spaces closed under taking finite (or arbitrary) topological products?

We end the section by showing that the class of UF spaces is closed under taking  $G_\delta$  subspaces. As with the class of MF spaces, this result cannot be extended to include  $F_\sigma$  subspaces. We first prove the result for open subspaces, which has a much simpler proof.

**Theorem 4.6.** The class of UF spaces is closed under taking open subspaces.

**Proof.** Let  $P$  be a poset and let  $U$  be an open subset of  $\text{UF}(P)$ . Let  $R$  be the set of all  $r \in P$  such that  $N_r \subseteq U$ . It can be seen that the map  $F \mapsto F \cap R$  is a homeomorphism from  $U$  to  $\text{UF}(R)$ .  $\square$

**Theorem 4.7.** The class of UF spaces is closed under taking  $G_\delta$  subspaces.

**Proof.** Let  $G_0$  be the space  $\text{UF}(P)$  for some poset  $P$  with order  $\prec_P$  and let  $G$  be a  $G_\delta$  subset of  $G_0$ . Then there is a descending sequence  $G_1, G_2, \dots$  of open subsets of  $G_0$  such that  $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$  and  $G = \bigcap_n G_n$ . Define

$$R = \{p \in P \mid p \in F \text{ for some } F \in \text{UF}(P) \cap G\}.$$

For each  $p \in R$ , let  $g(p) = \sup\{n \in \mathbb{N} \mid N_p \subseteq G_n\}$ , where  $g(p) = \infty$  if  $N_p \subseteq G$ . Define an order relation  $\prec_R$  on  $R$  by putting  $p \prec_R q$  if  $p \prec_P q$  and either  $g(q) < g(p) \leq \infty$  or  $g(q) = g(p) = \infty$ . We will show that the unbounded filters on  $(R, \preceq_R)$  are precisely the unbounded filters on  $P$  that are in  $G$  and do this by showing the following four claims.

*Claim 1:* Let  $F \in G \subseteq \text{UF}(P)$ ; then  $F$  is an unbounded filter in  $R$  under  $\preceq_R$ . By definition of  $R$ ,  $F \subseteq R$ . To show that  $F$  is a filter on  $R$ , fix  $p, q \in F$ . If  $g(p)$  or  $g(q)$  is infinite then  $p$  and  $q$  have a common extension  $r$  under  $\preceq_P$  with  $g(r) = \infty$ . Thus  $r$  is a common extension of  $p$  and  $q$  under  $\preceq_R$ . Otherwise, because  $F \in G$ , there is an  $r \in F$  with  $r \preceq_P p$ ,  $r \preceq_P q$  and  $N_r \subseteq G_{g(p)+g(q)+1}$ . Then  $g(r) > g(p) + g(q)$ ,  $r \prec_R p$  and  $r \prec_R q$ . As  $\preceq_R$  is a restriction of  $\preceq_P$ ,  $F$  is upward closed under  $\preceq_R$  and  $F$  is a filter in  $R$ . Furthermore,  $F$  must be unbounded in  $R$ , because a bound in  $R$  would also be in a bound  $P$ .

*Claim 2:* Let  $F \subseteq R$  be a filter in  $R$ ; then either  $\sup\{g(p) \mid p \in F\} = \infty$  or  $F$  is bounded. Suppose the supremum is  $n < \infty$  instead. There can only be one  $r \in F$  with  $g(r) = n$ , because  $F$  is a filter on  $R$ . Because  $r \in R$ , there is some  $F' \in \text{UF}(P)$  with  $r \in F'$  and  $F' \in G$ . Thus there is an  $r' \in F'$  with  $g(r') > g(r)$  and  $r' \prec_P r$ ; this means  $r' \prec_R r$ , which shows that  $F$  is bounded in  $R$ . This contradiction establishes the claim.

*Claim 3:* Let  $F$  be a bounded filter of  $P$  which is also a filter in  $R$ ; then  $F$  is bounded in  $R$ . Let  $r \in P$  be a lower bound for  $F$ . If  $N_r \not\subseteq G_n$  for some  $n$  then  $\sup\{g(p) \mid p \in F\} < n$  and  $F$  is bounded in  $R$  by Claim 2. Otherwise  $N_r \subseteq G$ , in which case  $r \in R$  and  $F$  is again bounded as a subset of  $R$ .

*Claim 4:* Let  $F$  be an unbounded filter in  $R$ ; then  $F$  is also an unbounded filter in  $P$ . To see this, consider the upward closure  $F'$  of  $F$  in  $P$ .  $F'$  is unbounded in  $P$ , by Claim 3. Claim 2 shows that  $F' \in G$ ; thus  $F' \subseteq R$ . The definition of  $F'$  shows that  $F \subseteq F'$ . Fix  $r \in F'$ ; then there must be a  $p \in F$  with  $p \preceq_P r$ . If  $g(p) = \infty$  then  $p \preceq_R r$  and so  $r \in F$ . Otherwise there must be a  $q \in F$  with  $q \preceq_P p$  and  $g(q) > g(r)$ . Then it follows from transitivity of  $\preceq_P$  and the definition of  $\prec_R$  that  $q \prec_P r$ ,  $q \prec_R r$  and  $r \in F$ . This shows  $F' = F$ .

Claims 1 and 4 show that the unbounded filters on  $R$  are exactly those unbounded filters on  $P$  which are in  $G$ . So the identity maps  $\text{UF}(P) \cap G$  to  $\text{UF}(R)$  and is surjective by Claim 4. It is straightforward to show that this map is a homeomorphism.  $\square$

## 5 Completeness properties

In this section, we establish that every poset space has the a completeness property known as the strong Choquet property. We then characterize the class of countably based MF spaces as precisely the class of second-countable  $T_1$  spaces with the strong Choquet property. We first establish a weaker property.

**Theorem 5.1.** Every poset space has the property of Baire.

**Proof.** Let  $X$  be  $\text{MF}(P)$  or  $\text{UF}(P)$ . Suppose that  $\langle U_i \mid i \in \mathbb{N} \rangle$  is a sequence of dense open sets in  $X$  and  $V$  is a fixed open set. We construct a sequence  $\langle p_i \mid i \in \mathbb{N} \rangle$  of elements of  $P$ . Let  $p_0$  be such that  $N_{p_0} \subseteq V \cap U_0$ . Given  $p_i$ , there is an unbounded or maximal filter in  $N_{p_i} \cap U_{i+1}$ . Choose  $p_{i+1}$  such that  $N_{p_{i+1}} \subseteq U_{i+1} \cap N_{p_i}$  and  $p_{i+1} \preceq p_i$ . In the end,  $F = \langle p_i \rangle$  is a linearly ordered subset of  $P$ . Thus  $F$  extends to an element of  $X$ . Clearly this element is in  $V \cap \bigcap_i U_i$ .  $\square$

We will now show that every poset space has the strong Choquet property, which is defined using a certain game. Let  $X$  be an arbitrary topological space. The strong Choquet game is the Gale–Stewart game (see [3] and [5]) defined as follows. The stages of play are numbered  $0, 1, 2, \dots$  and both players make a move in each stage. In stage  $i$ , player I plays an open set  $U_i$  and a point  $x_i$  such that  $x_i \in U_i$  and if  $i > 0$  then  $U_i \subseteq V_{i-1}$ . Then player II plays an open set  $V_i$  such that  $x_i \in V_i$  and  $V_i \subseteq U_i$ . At the end of the game, player I wins if  $\bigcap_i U_i$  is empty (or, equivalently, if  $\bigcap_i V_i$  is empty). Player II wins if  $\bigcap_i U_i$  is nonempty.

A space  $X$  has the *strong Choquet property* if player II has a winning strategy for the strong Choquet game on  $X$ . A winning strategy is a function that takes a position after player I has played and tells player II which open set to play, such that if player II follows the winning strategy then player II will always win the game regardless of what moves player I makes.

The strong Choquet property is strictly stronger than the property of Baire. Moreover, the class of topological spaces with the strong Choquet

property is closed under  $G_\delta$  subspaces and arbitrary topological products. It is known that the class of topological spaces with the property of Baire is not closed under binary products (an example is provided in [2]).

**Theorem 5.2.** Every poset space has the strong Choquet property.

**Proof.** We describe the strategy for player II informally. At the start of the game, player I plays an open set  $U_0$  and a point  $x_0$ . Player II translates the point  $x_0$  into a filter on  $P$ , then finds a basic neighbourhood  $q_0$  of  $x$  such that  $N_{q_0} \subseteq U_0$ . Player II then plays  $N_{q_0}$ . Now given  $\langle x_1, U_1 \rangle$  with  $x_1 \in N_{q_0}$ , Player II translates  $x_1$  to a filter on  $P$  and then finds a neighbourhood  $q_1$  of  $x_1$  such that  $q_1 \preceq_P q_0$  and  $N_{q_1} \subseteq U_1$ . Player II plays  $N_{q_1}$ . Player II continues this strategy, always choosing  $q_{i+1} \preceq_P q_i$ . At the end of the game, player II has determined  $\{q_i \mid i \in \mathbb{N}\}$ , a descending sequence of elements of  $P$ . This sequence extends to an element of  $X$  which is in  $\bigcap N_{q_i}$ . Player II has thus won the game.  $\square$

We use the strong Choquet property to obtain the following characterization of countably based MF spaces.

**Theorem 5.3.** A topological space is homeomorphic to a countably based MF space if and only if it is second countable,  $T_1$  and has the strong Choquet property.

We postpone the proof of this theorem temporarily to comment on the hypotheses involved in the characterization. Clearly, any space  $X$  homeomorphic to a countably based MF space must be  $T_1$  and second countable. We have already shown  $X$  must also have the strong Choquet property. Thus the new content of Theorem 5.3 is that the strong Choquet property is sufficient for a  $T_1$  second-countable space to be homeomorphic to a countably based MF space. In the non-second-countable setting, the strong Choquet property is not sufficient for a  $T_1$  space to be homeomorphic to an MF space.

**Theorem 5.4.** There is a Hausdorff strong Choquet space which is not homeomorphic to any MF space.

**Proof.** The space  $X$  consists of certain functions from  $\omega_1$  to  $\{0, 1\}$ . We put a function  $f$  in  $X$  if and only if there is an ordinal  $\alpha < \omega_1$  such that  $f(\beta) = 0$  for all  $\beta > \alpha$ . For each  $f \in X$  and each  $\alpha < \omega_1$ , the set  $\{g \in X \mid f(\beta) = g(\beta) \text{ for all } \beta < \alpha\}$  is declared to be an open set. The topology on  $X$  is the one generated by these open sets. It is clear that  $X$  is a Hausdorff space.

It is easy to show that  $X$  has the strong Choquet property. All that player II has to do is to play a basic open subset of the open set played by player I which contains also the point given by player I. In the end, the open sets played by player I in the countable number of rounds of the game and each round will fix countably many coordinates of a function in  $X$ . In the limit, countably many coordinates are fixed and we can find a point in the intersection of the sets played by I by forcing the remaining coordinates to map to 0.

We now show that  $X$  is not homeomorphic to any MF space. Suppose, by way of contradiction, that  $X \cong \text{MF}(P)$ . We construct a transfinite sequence  $\langle p_\alpha \mid \alpha < \omega_1 \rangle$  inductively. Let  $p_0$  be any basic open neighbourhood of the constant 0 function. Given  $\langle p_\alpha \mid \alpha < \beta \rangle$ , there is a first coordinate  $\gamma < \omega_1$  which is not fixed by any  $p_\alpha$ . Note that any intersection of countably many open sets in  $X$  is open. Thus we may choose  $p_\beta \in P$  such that  $p_\beta \preceq p_\alpha$  for all  $\alpha < \beta$  and  $f \in N_{p_\beta}$ . Choose any such  $p_\beta$ . At the end of this construction,  $\langle p_\alpha \mid \alpha < \omega_1 \rangle$  is linearly ordered and thus extends to a maximal filter  $F$ . Now the element of  $X$  corresponding to  $F$  sends uncountably many ordinals to 1, which is impossible.  $\square$

We now return to the proof of Theorem 5.3, which will occupy the remainder of this section. Let  $X$  be a fixed  $T_1$  space with a fixed countable basis and a fixed winning strategy for player II in the strong Choquet game. Our first step is to define a poset  $P$ . The elements of  $P$  are called *conditions*. A condition is a finite list of the form

$$\langle A, \pi_1, \pi_2, \dots, \pi_k \rangle$$

satisfying the following requirements.

- (1)  $A$  is a nonempty basic open set from the fixed countable basis. For each condition  $c$  we let  $S(c)$  denote the basic open set  $A$  appearing in  $c$ .
- (2) Each  $\pi_i$  is a finite (that is, partial) play of the strong Choquet game on  $X$  following the fixed winning strategy  $s_{\text{II}}$  for player II. We require each  $\pi_i$  to be of the form

$$\langle V_1, x_1, s_{\text{II}}(V_1, x_1), V_2, x_2, s_{\text{II}}(V_1, x_1, V_2, x_2), \dots, \\ V_r, x_r, s_{\text{II}}(V_1, x_1, V_2, x_2, \dots, V_r, x_r) \rangle.$$

Thus each  $\pi_i$  ends with an open set, which we will denote by  $U(\pi_i)$ . It is allowable that  $\pi$  is the empty sequence  $\langle \rangle$ , in which case  $U(\pi) = X$ .

(3) If a play  $\pi$  is in a condition then so is every initial segment of  $\pi$  that ends with a move by player II.

(4)  $A \subseteq U(\pi_i)$  for each  $i \leq k$ .

We define the order  $\prec$  on  $P$  as follows. Let  $c = \langle A, \pi_1, \pi_2, \dots, \pi_k \rangle$  and  $c' = \langle A', \pi'_1, \pi'_2, \dots, \pi'_l \rangle$  be any two conditions. We let  $c' \prec c$  if and only if

(5) For each finite play  $\pi_i$  in  $c$  there is a point  $x_n \in S(c)$  such that the longer play

$$\pi_i \frown \langle A, x_n, s_{\text{II}}(\pi_i \frown \langle A, x_n \rangle) \rangle$$

is in  $c'$ , that is, equals  $\pi'_j$  for some  $j \leq l$ .

(6)  $A' \subseteq A$  (this is actually a consequence of requirement (5)).

Requirement (3) in the definition of a condition allows us to prove that the order on  $P$  is transitive. Because each condition is finite, requirement (5) in the definition of the order relation ensures  $c \not\prec c$  for all  $c \in P$ . Thus  $\prec$  is a partial order on  $P$ .

**Lemma 5.5.** For any filter  $F$  on  $P$  the intersection  $\bigcap_{c \in F} S(c)$  is nonempty.

**Proof.** Let  $\langle A_i \mid i \in \mathbb{N} \rangle$  be an enumeration of all of the basic open sets which appear as  $S(c)$  for some  $c \in F$ ; here we are using the fact that  $X$  is second countable and that each  $S(c)$  is drawn from a fixed countable basis of  $X$ . It is clear that  $\bigcap_{c \in F} S(c)$  equals  $\bigcap_{i \in \mathbb{N}} A_i$ . We will show the latter intersection is nonempty.

We inductively construct a descending sequence of conditions  $\langle c_i \mid i \in \mathbb{N} \rangle$  and a sequence of finite plays  $\langle \pi_i \mid i \in \mathbb{N} \rangle$  so that  $\pi_{i+1}$  is an immediate extension of  $\pi_i$  for each  $i \in \mathbb{N}$ . At stage 0 let  $c_0$  be any condition in  $F$  such that  $S(c_0) = A_0$  and let  $\pi_0$  be any finite play in  $c_0$ .

At stage  $i + 1$  let  $c$  be any condition in  $F$  such that  $S(c) = A_i$ . Let  $c_{i+1}$  be a common extension of  $c$  and  $c_i$  in  $F$ . It is clear that  $S(c_{i+1}) \subseteq S(c) = A_i$ . Choose  $\pi_{i+1}$  to be any play in  $c_{i+1}$  which is an immediate extension of  $\pi_i$ .

Now assume the entire sequence  $\langle \pi_i \rangle$  has been constructed. These partial plays determine an infinite play  $\gamma$  of the strong Choquet game following the strategy for player II. Thus the intersection of the open sets played by player I in  $\gamma$  is nonempty. By construction, each set  $A_i$  has a subset played by player I at some stage of  $\gamma$ . Thus  $\bigcap_i A_i$  is nonempty.  $\square$

**Lemma 5.6.** Let  $c_1$  and  $c_2$  be two conditions and let  $x \in S(c_1) \cap S(c_2)$ . There is a condition  $c$  such that  $c \prec c_1$ ,  $c \prec c_2$  and  $x \in S(c)$ .

**Proof.** Begin by letting  $c$  be empty. For each  $\pi$  in  $c_1$  we put the longer play

$$\pi \frown \langle S(c_1), x, s_{\text{II}}(\pi \frown \langle S(c_1), x \rangle) \rangle$$

into  $c$ . For each  $\pi$  in  $c_2$  we put

$$\pi \frown \langle S(c_2), x, s_{\text{II}}(\pi \frown \langle S(c_2), x \rangle) \rangle$$

into  $c$ . For each  $\pi$  that has been added to  $c$  we add all initial segments of  $\pi$  ending with a move by player II. We then let  $S(c)$  be a basic open neighbourhood of  $x$  which is a subset of the open set  $\bigcap_{\pi \in c} U(\pi)$ . This construction ensures that  $c$  is a condition satisfying the conclusions of the lemma.  $\square$

**Lemma 5.7.** Let  $F$  be a maximal filter on  $P$ . The intersection  $\bigcap_{c \in F} S(c)$  contains a single point.

**Proof.** By Lemma 5.5 we know that  $\bigcap_{c \in F} S(c)$  is nonempty. Suppose that  $x, y$  are distinct points in  $\bigcap_{c \in F} S(c)$ . Let  $A$  be a basic open neighbourhood of  $x$  such that  $y \notin A$ . We construct an filter  $G$  inductively. At stage  $n$  we construct  $G_n \subseteq P$  and in the end we let  $G$  be the upward closure of  $\bigcup_n G_n$ . To begin, let  $G_0 = F \cup \{\langle A, \langle \rangle \rangle\}$ . At stage  $i + 1$ , we know by induction that  $x \in S(c)$  for every  $c \in G_i$ . Thus we can apply Lemma 5.6 repeatedly so that  $G_i \subseteq G_{i+1}$ , every pair of conditions in  $G_i$  has a common extension in  $G_{i+1}$  and  $x \in S(c)$  for every  $c \in G_{i+1}$ .

It is immediate from the construction that  $G = \bigcup_i G_i$  is a filter which properly extends  $F$ . This shows that  $F$  was not maximal.  $\square$

**Proof of Theorem 5.3.** For each  $F \in \text{MF}(P)$  we denote the single point in  $\bigcap_{c \in F} S(c)$  by  $\Phi(F)$ . We show that  $\Phi$  is a homeomorphism from  $\text{MF}(P)$  to  $X$ .

We first show that  $\Phi$  is an injective map. Suppose that  $F$  and  $F'$  are maximal filters on  $P$  such that  $x \in \bigcap_{c \in F} S(c)$  and  $x \in \bigcap_{c \in F'} S(c)$ . By following a procedure similar to the proof of Lemma 5.7 we may find a filter  $G$  such that  $F \subseteq G$  and  $F' \subseteq G$ . Thus, by maximality, we have  $F = F' = G$ .

We next show that  $\Phi$  is a surjective map. Let  $x \in X$  be fixed. Let  $\langle A_i \mid i \in \mathbb{N} \rangle$  be a sequence of basic open sets such that  $\bigcap_i A_i = \{x\}$ . The existence of this sequence requires that  $X$  be  $T_1$  and first countable. For

each  $i \in \mathbb{N}$  let  $c_i = \langle A_i, \langle \rangle \rangle$ . Following a method similar to the proof of Lemma 5.7, we can construct a filter  $F$  such that  $c_i \in F$  for each  $i \in \mathbb{N}$ . Let  $G$  be an extension of  $F$  to a maximal filter. Now  $S = \bigcap_{c \in G} S(c)$  is nonempty by Lemma 5.5 and  $S \subseteq \bigcap_i A_i = \{x\}$  by construction, so  $\Phi(G) = x$ .

It remains to show that  $\Phi$  is open and continuous. This follows from Lemma 5.6; for each  $x \in X$  and each condition  $c$ , we have

$$c \in \Phi^{-1}(x) \Leftrightarrow x \in S(c).$$

This shows that  $X$  is homeomorphic to  $\text{MF}(P)$ . By Theorem 3.2, we may find a countable subposet  $R$  of  $P$  such that  $X$  is homeomorphic to  $\text{MF}(R)$ . This completes the proof.  $\square$

## 6 An application to domain theory

In this section, we apply the characterization of countably based MF spaces to characterize those second-countable spaces with a domain representation. Our result gives a complete solution to the so-called model problem for second-countable spaces in domain theory.

A domain is a certain type of poset (defined below) and every domain is a topological space with a topology known as the Scott topology. A domain representation of a topological space  $X$  is a domain  $D$  such that  $X$  is homeomorphic to the topological space consisting of the maximal elements of  $D$  with the relative Scott topology. The history of such representations is thoroughly described by Martin [7]. It is known that every complete separable metric space has a domain representation (see Lawson [6]) and that every space with a domain representation is  $T_1$  and has the strong Choquet property (Martin [7]). We now show that the strong Choquet property is sufficient for a  $T_1$  second-countable space to have a domain representation.

We summarize the definitions from domain theory that we require; these definitions are explored fully by Giertz *et al.* [4]. A subset  $I$  of a poset  $\langle P, \preceq \rangle$  is *directed* if every pair of elements in  $I$  has an upper bound in  $I$ . A poset  $P$  is said to be a d.c.p.o. if every directed subset of  $P$  has a least upper bound. Any d.c.p.o.  $D$  has a second order relation  $\ll$ , known as the *way below* relation, under which  $q \ll p$  if and only if whenever  $I \subseteq D$  is a directed set with  $p \leq \sup I$  there is some  $r \in I$  with  $q \preceq r$ . For each  $p \in D$  we put  $\Downarrow p = \{q \in D \mid q \ll p\}$  and  $\Uparrow q = \{p \in D \mid q \ll p\}$ . A d.c.p.o.  $D$  is *continuous* if  $\Downarrow p$  is directed and the equality  $p = \sup \Downarrow p$  holds

for every  $p \in D$ . A *domain* is a continuous d.c.p.o. A subset  $B$  of a domain  $D$  is a *basis* if  $B \cap \downarrow p$  is directed and  $p = \sup(B \cap \downarrow p)$  for every  $p \in D$ . A domain is  $\omega$ -*continuous* if it has a countable basis. An element  $p$  of a d.c.p.o. is *compact* if  $p \ll p$ . A d.c.p.o.  $D$  is  $\omega$ -*algebraic* if the compact elements of  $D$  form a countable basis for  $D$ . The *Scott topology* on a d.c.p.o.  $D$  is generated by the basis  $\{\uparrow p \mid p \in D\}$ . A *domain representation* of a space  $X$  is a homeomorphism between  $X$  and the maximal elements of a domain with the Scott topology.

**Theorem 6.1.** A topological space has a domain representation via an  $\omega$ -algebraic d.c.p.o. if and only if the space is second-countable,  $T_1$  and has the strong Choquet property.

**Proof.** It can be seen that any space with a domain representation satisfies the  $T_1$  separation property and a result of Martin [7] shows that any space with a domain representation has the strong Choquet property. Therefore, we only need to prove that a second-countable  $T_1$  strong Choquet space has a domain representation via an  $\omega$ -algebraic d.c.p.o. We use the following lemma, which follows easily from the definitions.

**Lemma 6.2.** Suppose that  $P$  is a countable poset. The set of all filters on  $P$ , ordered by inclusion, is an  $\omega$ -algebraic d.c.p.o.  $D$ . The maximal filters on  $P$  are precisely the maximal elements of  $D$  and the compact elements of  $D$  are precisely the principal filters on  $P$ . Moreover, the poset topology on  $\text{MF}(P)$  corresponds exactly to the Scott topology on the maximal elements of  $D$ .

We showed in Theorem 5.3 that any second-countable  $T_1$  strong Choquet space is homeomorphic to  $\text{MF}(P)$  for a countable poset  $P$ . It follows immediately from the lemma above that such a space also has a domain representation via an  $\omega$ -algebraic d.c.p.o.  $\square$

The next corollary follows from the fact that any space with a domain representation is  $T_1$  and has the strong Choquet property.

**Corollary 6.3.** If a second-countable space has a domain representation then it has a representation via an  $\omega$ -algebraic d.c.p.o.

A *Scott domain* is a domain in which every pair of elements with an upper bound has a least upper bound. Lawson [6] has shown that any space with a domain representation via a countably based Scott domain is a complete

separable metric space. It can be seen that posets constructed in Theorem 5.3 do not, in general, give Scott domains, even when the posets are constructed from complete separable metric spaces.

## 7 Semi-Topogenous Orders

In this section, we prove results which give a partial solution to the question of which arbitrary (not necessarily second countable) topological spaces are homeomorphic to MF spaces.

When a topological space  $X$  is homeomorphic to  $\text{MF}(P)$ , it is not necessary that the elements of  $P$  are of subsets of  $X$ . If we replace each element of  $P$  by the basic open subset of  $X$  to which it is homeomorphic, the poset order on  $P$  will determine an order relation on the basic open subsets of  $X$  corresponding to these poset elements. It is natural to ask whether the existence of a suitable order relation on a collection of subsets is sufficient for a topological space to be homeomorphic to an MF space.

Császár [1] considered many different types of orders and their connections to topology. The basic concept is that of a semi-topogenous order.

**Definition 7.1.** A *semi-topogenous order* is a binary relation  $\sqsubset$  on the powerset of a topological space  $X$  satisfying the following axioms for all  $u, v, w \subseteq X$  [1, Chapter 2]:

- $\emptyset \sqsubset \emptyset$  and  $X \sqsubset X$ ;
- $v \sqsubset w \Rightarrow v \subseteq w$ ;
- $u \subseteq v \sqsubset w \Rightarrow u \sqsubset w$ ;
- $u \sqsubset v \subseteq w \Rightarrow u \sqsubset w$ .

Császár considered orders which are only linked to topology, such as the order which says that  $w$  is a neighbourhood of  $v$ , that is, that  $v$  is not near to  $X \setminus w$ . It might happen that some but not all open supersets  $w$  of a given set  $v$  satisfy  $v \sqsubset w$ . Nevertheless, although this is not made explicit by Császár, it is quite convenient to postulate also a connection between the topology and the open spaces.

Recall that the open kernel of a set is the union of all its open subsets. We say that the topological space  $X$  is *generated* by the order  $\sqsubset$  if for each

$u \subseteq X$  the set  $\bigcup\{o \subseteq X \mid o \sqsubset u\}$  is the open kernel of  $u$ . In this case a set  $w$  is open if and only if it is the union of all  $v$  such that  $v \sqsubset w$ . It follows that if  $v \sqsubset w$  then there is an open  $o$  with  $v \subseteq o \subseteq w$ ; the converse of this last implication does not always hold. Every topological space is generated by some semi-topogenous order, for one can define  $v \sqsubset w$  to hold if and only if there is an open set  $o$  with  $v \subseteq o \subseteq w$ .

**Remark 7.2.** There is a close relationship between semi-topogenous orders and the way below relation  $\ll$  on a continuous d.c.p.o., which was discussed in Section 6. The following true properties of the way below relation are obtained by dualizing the second, third and fourth properties in the definition of a semi-topogenous order:

$$\begin{aligned} v \ll w &\Rightarrow v \leq w \\ u \leq v \ll w &\Rightarrow u \ll w \\ u \ll v \leq w &\Rightarrow u \ll w \end{aligned}$$

The fact that these are dual forms follows from the fact that points in a topological space are minimal as nonempty subsets under  $\subseteq$  but are maximal elements of a domain representing the topological space; for this reason, we write  $\leq$  for  $\supseteq$  and  $\ll$  for  $\sqsubset$ . The requirement that  $\bigcup\{o \mid o \sqsubset u\}$  is the open kernel of  $u$  corresponds exactly to the fact that  $\{x \mid y \ll x\}$  is the open kernel of an element  $y$  of a continuous d.c.p.o. with the Scott topology.

Thus if a space  $X$  has a representation via a continuous d.c.p.o.  $D$  then the dual of the way below relation on  $D$  is a semi-topogenous order (except that it is defined only on a subset of the powerset of  $X$ ) which generates the topology on  $X$ . Semi-topogenous orders can be viewed as a generalization of the way below relation which is applicable to the case when the d.c.p.o. is the full powerset of a topological space. It appears that semi-topogenous orders are related to auxilliary relations as defined by Giertz *et al.* [4], although a formal relationship seems difficult to state.

A *filter* in a topological space  $X$  is a collection of nonempty subsets that is closed under finite intersection and under superset. A filter *has an open basis* if for every  $w$  there is a  $v$  in the filter with  $v \subseteq w$ . As in general there need not be a point contained in every element of a filter, we are interested in a condition on filters that requires their sets to contain a common point. Our condition that a filter meets a semi-topogenous order will imply that this filter has also an open basis, while a completeness condition will ensure that each filter that meets the order has a nonempty intersection.

**Definition 7.3.** Let  $X$  a space with a semi-topogenous order  $\sqsubset$  generating its topology. A filter  $U$  on  $X$  *meets*  $\sqsubset$  if for every  $w \in U$  there is a  $v \in U$  with  $v \sqsubset w$ .  $X$  is *complete for*  $\sqsubset$  if for every filter  $U$  in the space  $X$  which meets  $\sqsubset$  there is a point  $x$  with  $x \in u$  for all  $u \in U$ .

**Theorem 7.4.** Let  $X$  be a  $T_1$  space with a semi-topogenous order  $\sqsubset$  generating its topology such that  $X$  is complete for  $\sqsubset$ . Then  $X$  is homeomorphic to an MF space.

**Proof.** Let  $P$  consist of the nonempty open subsets of  $X$  and let  $p \prec q$  if and only if  $p \neq q$  and  $p \sqsubset q$ . The relation  $\prec$  is obviously transitive and antireflexive, and so it makes  $P$  into a poset.

For each  $x \in X$  let  $U_x$  be the set of all  $p \in P$  with  $\{x\} \sqsubset p$ . If  $p, q \in U_x$  then the open kernel  $u$  of  $p \cap q$  contains  $x$  and thus there is an  $r \sqsubset u$  with  $x \in r$ . As the open kernel of  $r$  again contains  $x$ ,  $\{x\} \sqsubset r$ . So  $r \in U_x$ ,  $r \sqsubset p$  and  $r \sqsubset q$ . Thus  $U_x$  is a filter on  $P$ .

If  $V$  is a maximal filter on  $\text{MF}(P)$  then  $V$  also meets  $\sqsubset$ . If  $v$  generates  $V$  then  $v$  is open (by definition) and not empty. For every  $x \in v$  there is a  $w \sqsubset v$  with  $x \in w$ ; by maximality  $w = v$ . Thus  $v \sqsubset v$  and every  $w \subseteq X$  with  $v \subseteq w$  satisfies  $v \sqsubset w$  and  $w \in V$ . If there is no single element generating  $V$  then there is for every  $v, w \in V$  an  $u \in V$  with  $u \prec v$  and  $u \prec w$ . Then it follows that  $u \sqsubset v$  and  $u \sqsubset w$ . Furthermore, there is an  $t \prec u$  with  $t \in V$ ; then it follows that  $t \sqsubset v \cap w$ . So  $V$  contains all supersets of  $v \cap w$  and thus  $V$  is a filter. Furthermore,  $V$  meets  $\sqsubset$ .

So by assumption there is a point  $x$  contained in all sets of  $V$ . Thus  $V \subseteq U_x$  and by the maximality of  $V$ ,  $V = U_x$ .

Therefore, every filter  $U_y$  is contained in a filter  $U_x$  which is maximal. Due to the  $T_1$  property,  $y = x$ ; otherwise  $U_y$  would contain a  $p$  with  $x \notin p$  in contradiction to the fact that  $U_y \subseteq U_x$ .

So, the mapping  $x \mapsto U_x$  is a bijection from  $X$  to the maximal filters on  $P$  and it is easy to see that this mapping is indeed a homeomorphism.  $\square$

We do not know whether every MF space has a semi-topogenous order satisfying the hypotheses of the previous theorem. We have established the following partial result.

**Theorem 7.5.** If  $X = \text{MF}(P)$  and  $P$  satisfies

$$\forall p, q, r [p \prec q \wedge N_q \subseteq N_r \Rightarrow p \prec r] \quad (1)$$

then there is a semi-topogenous order  $\sqsubset$  generating the topology of  $X$  such that  $X$  is complete for  $\sqsubset$ .

**Proof.** For any  $v, w \subset X$ , let  $v \sqsubset w$  if either  $v = \emptyset$ ,  $w = X$ , there is an open atom  $u$  with  $v \subseteq u \subseteq w$  or there are  $p, q \in P$  with  $v \subseteq N_p$ ,  $p \prec q$  and  $N_q \subseteq w$ . Note that  $N_p \subseteq N_q$  in the last case.

It is easy to see that  $\sqsubset$  is a semi-topogenous order. We next show that  $\sqsubset$  generates the topology of  $X$ . Let  $w$  be an open set and  $x$  be a point in  $w$ . There is an open set  $N_q$  with  $\{x\} \subseteq N_q \subseteq w$ . In the case that  $\{x\} = N_q$ ,  $N_q \sqsubset w$ . In the case that  $\{x\} \neq N_q$  there is a further  $p \prec q$  with  $x \in N_p$ : The reason is that given an  $y \in N_q \setminus \{x\}$ , the maximal filter  $U_x$  belonging to  $x$  must contain a  $p \prec q$  which does not contain  $y$  by the  $T_1$  axiom. Then  $\{x\} \subseteq N_p \sqsubset w$ . So  $w$  is the union of all  $v$  with  $v \sqsubset w$ .

Now let  $W$  be a filter in the topological space  $X$  which meets  $\sqsubset$ . If  $W$  contains an  $r$  such that  $N_r$  is atomic, that is, a singleton  $\{x\}$ , then every  $u \in W$  contains  $x$  since otherwise  $N_r \cap u = \emptyset$  in contradiction to  $W$  being a filter.

If  $W$  does not contain an  $r$  such that  $N_r$  is atomic, then let  $V$  be the set of all  $p \in P$  such that  $N_p \in W$ . Given any  $p, q \in V$  there is an  $u$  such that  $u \sqsubset N_p \cap N_q$ . So there are  $r, t$  with  $u \subseteq N_r$ ,  $N_t \subseteq N_p \cap N_q$  and  $r \prec t$ . It follows that  $r \prec p$  and  $r \prec q$ . Thus  $V$  is the basis of a filter on  $P$ ; this filter is contained in a maximal filter on  $P$  which is of the form  $U_x$  for some point  $x \in X$ . This  $x$  is then in  $N_p$  for all  $p \in V$ . Let  $u \in W$ . As  $W$  meets  $\sqsubset$ , there is a  $p \in V$  with  $N_p \subseteq u$ . It follows that  $x \in N_p$  and  $x \in u$ . So  $x$  is a common point of the sets in  $W$ .  $\square$

The two posets constructed in Theorems 2.1 and 2.2 satisfy condition (1) and thus they are examples of a poset space that is complete for a semi-topogenous order generating its topology.

**Example 7.6.** For every complete metric space and every locally compact Hausdorff space there exists a semi-topogenous order  $\sqsubset$  which generates the topology of  $X$  and for which  $X$  is complete.

**Remark 7.7.** Assume that  $X$  is a space which is complete for a semi-topogenous order generating its topology. Then one can not only show that  $X$  is homeomorphic to an MF space but also that the winning strategy for player II is quite easy. Given any open set  $u$  and any point  $x \in u$  by player I, player II only has to choose an open  $v$  with  $\{x\} \subseteq v \sqsubset u$ . It does not matter

which  $v$  with this condition is chosen and the history of the game can be ignored. The result of the construction will be, at the end of the game, a basis for a filter which meets  $\sqsubset$  and thus this filter has a common point.

This shows that the neighbourhood spaces that we consider here satisfy a restricted version of the strong Choquet property. The intuition behind this restriction is that one wishes to study non-second-countable spaces by considering “transfinite games.” The role of player I is replaced by considering filters instead of descending sequences, and the winning strategy of player II is reduced to a neighbourhood relation  $\sqsubset$  which could be interpreted as saying that if  $\{x\} \subseteq v \sqsubset u$  then  $v$  is a good move for player II.

Indeed, the notion of completeness of spaces with respect to a semi-topogenous order  $\sqsubset$  is based on this idea: Let the strategy of player II be just to follow  $\sqsubset$  and let player I build a filter  $U$  such that for every  $w \in U$  there is a  $v \in U$  which player II might have chosen as a response to  $w$ , that is, a  $v \sqsubset w$ ; then the intersection of all  $u \in U$  is not empty.

## 8 Cardinality of poset spaces

In this section, we establish perfect set theorems for countably based Hausdorff poset spaces. These theorems show that these spaces are either countable or have the cardinality  $2^{\aleph_0}$  of the continuum.

**Theorem 8.1.** Any countably based Hausdorff poset space has either countably many points or has cardinality  $2^{\aleph_0}$ .

**Corollary 8.2.** Any countably based Hausdorff poset space has either countably many points or contains a perfect closed set.

**Proof.** Any second-countable Hausdorff space of cardinality  $2^{\aleph_0}$  contains a perfect closed set. The complement of the perfect closed set is the union of all the basic open sets from a fixed countable basis which contain fewer than  $2^{\aleph_0}$  points.  $\square$

To prove Theorem 8.1, we introduce a class of Gale–Stewart games. These games are inspired by the  $*$ -games in descriptive set theory (as described in [5]). For each poset  $P$  we define a game which we call the *poset star game on  $P$* . There are two players. The play proceeds in stages numbered  $0, 1, 2, \dots$ . At stage  $t$ , player I plays a pair  $\langle p_1^t, p_2^t \rangle \in P \times P$ . Player II plays

a number  $n_t \in \{1, 2\}$ . Player I wins the game if the following conditions hold for all  $t$ .

- (1)  $p_1^t \perp p_2^t$ .
- (2)  $p_1^{t+1} \preceq p_{n_t}^t$  and  $p_2^{t+1} \preceq p_{n_t}^t$ .

Player II wins if player I does not win; there are no ties.

A *strategy* for a player is a function that tells the player what to do at any possible move of the game. The strategy is a *winning strategy* if the player will win any play of the game in which the player uses the strategy to choose every move. It is impossible for both players to have a winning strategy for the same game.

**Lemma 8.3.** Let  $P$  be a poset. Either player I or player II has a winning strategy for the poset star game on  $P$ .

**Proof.** The set of infinite plays of the poset star game on  $P$  that are winning for player I is a closed subset of all possible plays of the game. The proof follows from a theorem of Gale and Stewart known as closed determinacy.  $\square$

**Lemma 8.4.** Suppose that  $X$  is a Hausdorff poset space based on a countable poset  $P$  and player I has a winning strategy for the poset star game on  $P$ . Then  $X$  has cardinality  $2^{\aleph_0}$ .

**Proof.** It suffices to prove the result for  $\text{MF}(P)$ , which is a subset of  $\text{UF}(P)$ . Let  $s_I$  be a winning strategy for player I and let  $f \in \{0, 1\}^{\mathbb{N}}$ . Consider the play in which player I follows  $s_I$  while player II uses  $f$  as a guide; that is, player II plays  $f(n)$  at stage  $2n$ . Because  $s_I$  is a winning strategy for player I, this play determines a descending sequence  $F(f)$  of elements of  $P$ . This sequence extends to a maximal filter. For distinct  $f, g \in \{0, 1\}^{\mathbb{N}}$  the sequences  $F(f)$  and  $F(g)$  contain incompatible elements and thus cannot extend to the same filter. Therefore the space  $\text{MF}(P)$  has cardinality  $2^{\aleph_0}$ .  $\square$

**Lemma 8.5.** Let  $X$  be a countably based Hausdorff poset space based on the poset  $P$ . If player II has a winning strategy for the poset star game on  $P$  then  $X$  is countable.

**Proof.** Let  $s_{II}$  be a winning strategy for player II. We say that a finite play  $\sigma$  of length  $2k$  is *compatible with*  $s_{II}$  if  $s_{II}(\sigma[2i+1]) = \sigma(2i+2)$  whenever  $2i+2 \leq k$ . We say that a play  $\sigma$  of even length is a *good play* for a point  $x$

if  $\sigma$  is compatible with  $s_{II}$  and  $x$  is in the open set chosen by player II in the last move of  $\sigma$ . A good play for  $x$  is a *maximal play* if it cannot be extended to a longer good play for  $x$ ; this means that no matter what pair of disjoint open sets player I plays,  $s_{II}$  will direct player II to choose an open set not containing  $x$ .

If player II has a winning strategy then every point  $x$  has a maximal play. Note that the empty play is trivially a good play for  $x$ . If every good play for  $x$  could be extended to a larger good play for  $x$ , then it would be possible for player I to win the game by always leaving the game in a position that is good for  $x$ . This play of the game would follow  $s_{II}$ , a winning strategy for player II, which is a contradiction.

If  $\sigma$  is a good play for two points  $x$  and  $y$  then  $\sigma$  is not a maximal play for both  $x$  and  $y$ . For player I could play  $\langle U_1, U_2 \rangle$  in response to  $\sigma$ , where  $x \in U_1$ ,  $y \in U_2$  and  $U_1 \cap U_2 = \emptyset$ . Here we are using the assumption that the topology of  $X$  is Hausdorff.

We have now shown that every point in the  $X$  has a maximal play and that no play is maximal for two points. Since the set of maximal plays is countable, this implies that the set of points in  $X$  is countable.  $\square$

We remark that the statement “Every closed subset of a countably based Hausdorff MF space is either countable or has a perfect closed subset” is independent of ZFC set theory; this result is established in [9].

## Acknowledgments

We would like to thank the Institute for Mathematical Sciences at the National University of Singapore for organizing the wonderful Computational Prospects of Infinity workshop in 2005 which made this work possible. We also thank Steffen Lempp and Sasha Rubin for thoughtful comments. We would like to thank Jimmie Lawson and Ralph Koppermann for their helpful comments on the domain theory results in Section 6.

Some of the results presented here appeared in the first author’s PhD thesis [8], supervised by Stephen Simpson at the Pennsylvania State University.

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