On the foundation of stability*

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This Version: September 2006

Abstract

The concept of stability *á la* J. von Neumann and O. Morgenstern, which is composed of a pair of internal and external stability requirements, formalizes the idea of standard of behavior. This paper studies the decision-theoretic foundation of stability, by establishing some epistemic conditions for a "stable" pattern of behavior in the context of strategic interaction.
JEL Classification: C70, C71, D81.

Keywords: game theory, stable sets, rationality, common knowledge

*This paper is based upon part of my manuscript entitled "Information, Knowledge, and Stability." The earlier version was presented at the Conference on Current Trends in Economics, Rodos, Greece; the International Conference on Game Theory, Stony Brook, NY; the Far Eastern Meeting of the Econometric Society, Kobe, Japan; and the ICM Satellite Conference in "Game Theory and Applications," Qingdao, China. This paper partially done while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2005.

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1. Introduction

In their classics, J. von Neumann and O. Morgenstern (1944) enthusiastically advocated the idea of stability by introducing a solution concept of the vN-M stable set which was interpreted as a "standard of behavior." Ever since then, the criterion of stability has been widely applied in economics and other social sciences. For example, the criterion is applied to the political analysis of voting behavior and the economic analysis of the formation of cartels in large markets (see Hart (1974)). See Lucas (1994) and Shubik (1982) for extensive surveys.

Greenberg (1990) advanced this line of research by providing an integrative approach to the study of formal models in the social and behavioral sciences using the unique criterion of stability. Most remarkably, this approach unifies the analysis of cooperative and noncooperative games,[1] and revitalizes this old idea in game theory; see, e.g., Arce (1994), Chwe (1994), Diamantoudi (2003, 2005), Ehlers (2007), Einy *et al.* (1996), Einy and Shitovitz (1996, 1997, 2003), Ferreira (2001), Greenberg (1989a, 1989b, 1992), Greenberg *et al.* (1996, 2002), Greenberg and Shitovitz (1994), Huang and Luo (2006), Inarra *et al.* (2006), Luo (2001, 2006), Luo and Ma (2001), Mariotti (1997), Nakanishi (1999, 2001), Oladi (2004, 2005), Ray (1998), Rosenmüller and Shitovitz (2000), Shitovitz (1994), Shitovitz and Weber (1997), Suzuki and Muto (2005), Tadelis (1996), and Xue (1997, 1998, 2000, 2002) for some interesting applications in game theory and economics.

While von Neumann and Morgenstern first defined the vN-M stable set on imputations, they referred this idea to a variety of social organizations (see von Neumann and Morgenstern 1944, Sections 4.6, 4.7, and 65.1). More specifically, consider an abstract game $(X, \succ)$ consisting of a set X of outcomes and of a dominance relation $\succ$ over outcomes. A *vN-M stable set* is defined as a subset $\mathcal{K} \subseteq X$ that satisfies:

1. [internal stability] no y in $\mathcal{K}$ is dominated by an x in $\mathcal{K}$, and
2. [external stability] every y not in $\mathcal{K}$ is dominated by some x in $\mathcal{K}$.

[1] Kalai and Schmeidler (1977) also offered a unified "admissible set" approach to cooperative and noncooperative games. I am grateful to Prof. Kalai for drawing my attention to this. See Deemen (1991) and Delver and Monsuur (2001) for other generalisations of the von Neumann-Morgenstern stable set.

The definition of a stable set formalizes a “standard of behavior,” and is bounded into two requirements on a standard of behavior: on the one hand, it is free of inner contradictions: what is expected to be followed is not refuted by the outcomes that the standard of behavior generates. On the other hand, it is immune from external inconsistencies: the outcomes not governed by the standard of behavior can be overruled within the limits of the accepted standard of behavior. Although the stability criterion appears to be methodologically profound, conceptually sophisticated, theoretically elegant, and applicably fruitful, to the best of our knowledge no formal foundation has been laid in the literature. Up until now, most theorists simply took this criterion as a normative requirement.[2] The purpose of this paper is to study the decision-theoretic foundation for the criterion of stability.

The major reason for pursuing the study of this paper is the following. To connect with real-life phenomena, von Neumann and Morgenstern (1944, pp.40-43) interpreted a vN-M stable set as the “established order of society” or “accepted standard of behavior.” As nicely put by Shubik (1982, p.161), “a stable set [can] be viewed as a *standard of behavior* – or a *tradition, social convention, canon of orthodoxy*, or *ethical norm*.” Following this line of interpretation, the notion of stability is conceptually distinct from that of rationality in economics since they belong to two different disciplines, *homo economicus* and *homo sociologicus*, respectively (see Fig.).

	homo economicus	*homo sociologicus*
Pioneer	Adam Smith	Emile Durkheim
Terminology	instrumental rationality (cf. Hume’s *Treatise on Human Nature*)	social norm
Major Feature	if you want to achieve $\mathfrak{z}$, do $\mathfrak{x}$	do $\mathfrak{x}$; or don’t do $\mathfrak{x}$

Fig. *homo economicus* vs. *homo sociologicus*.

[2] As noted by Aumann (1987, pp.58-59), “It is not in any sense obvious from the definition [of a stable set]; indeed, there is no hint of organizational forms in the definition – the definition refers only to outcomes. The conceptual connection with organizational forms is in the interpretation only and has no formal status; and it is a matter of some mystery that this particular definition should lead to so frequently and consistently to this kind of interpretation.”

Whereas sociologists view behavior as driven by social norms and other macro forces, economists instead insist that all behavior should be explained in terms of rationality. Once we accept the basic tenet of methodological individualism in economics, it naturally leads to the fundamental questions: Why should economists be interested in stability? What is its microfoundation? And under which conditions is such a stable pattern of behavior sustained? In this paper we investigate these questions by establishing epistemic conditions for a "stable" pattern of behavior in the context of strategic interaction.

In recent years a strand of literature has emerged that explores the solutions in noncooperative games from a decision-theoretic viewpoint – i.e., in terms of rationality and epistemic states. It is thus an important research subject to extend this line of research to the criterion of stability in this direction. Inspired primarily by the work of Aumann (1995) and Aumann and Brandenburger (1995), this paper investigates the decision-theoretic foundation for the notion of stability within the standard semantic framework; see, e.g., Aumann (1976, 1987) and Osborne and Rubinstein (1994, Chapter 5).

Some salient features in the set-up of this paper are as follows.

Feature A. In the context of a game, this paper utilizes a generalization of the vN-M stable set proposed by Luo (2001). The criterion of stability is thus represented by this generalized concept of the vN-M stable set (see Definition 1).

Feature B. To deal with the set-valued solution concept, this paper employs a variant of the semantic knowledge model. Contrary to the standard model, at any state of the world each player is associated with a nonempty subset of strategies that is interpreted as a choice set in orthodox choice theory, i.e., the elements in a choice set are regarded as the most desired choices.[3] In the literature Samuelson (1992) used a similar model

[3] To methodologically understand a knowledge model, it is very helpful to refer to Aumann and Brandenburger's (1995) insightful and penetrating discussions. As they wrote, "[It] is not a prescriptive model; it does not suggest actions to the players. Rather, it is a formal framework – a language – for *talking* about actions, payoffs, and beliefs. For example, it enables us to say whether a given player is behaving rationally at a given state, whether this is known to another player, and so on ...... The 'problem-solution' viewpoint is the older, classical approach of game theory. The viewpoint adopted here is different – it is *descriptive.* Not *why* the players do what they do, not what *should* they do; just what *do* they do, what *do* they believe ... Not 'why,' not 'should,' just *what.* Not that i does a *because* he believes b; simply that he does a, and believes b" (Aumann and Brandenburger 1995, pp.1174-1175).

of knowledge to discuss the "common knowledge of admissibility;" see, also, Basu and Weibull (1991), Borgers and Samuelson (1992), and Ewerhart (1998).

Feature C. This paper uses a set version of (Bayesian) rationality in economics. Specifically, a player is rational at a state if the player's choice set consists precisely of the strategies that maximize his expected utility given his information at that state.

In spite of the conceptual difference between stability and rationality, this paper presents some formal foundation for the notion of stability by identifying exact assumptions about rationality, knowledge, and information that lie behind the set-valued solution of the stable set. In this paper we formulate and prove the following results:

1. Achieving common knowledge of rationality (CKR) implies an externally stable set which, in turn, is contained in a commonly-known internally stable set. Furthermore, the event CKR is itself associated with a stable set if it is closed under rational behavior (see Theorem 1(i)).

2. In the case of a "linear" structure state space, achieving CKR implies a stable set (see Theorem 2); rationality alone can imply a stable set whenever the scope of choices is mutually known (see Theorem 3).

3. Any commonly-known stable set implies CKR (see Theorem 1(ii)); and any stable set can be attained in terms of CKR (see Theorem 4).

The rationality behind stability is found to be abundant in strategic settings, so that this paper provides some theoretical justification for the analysis of the "stable" pattern of behavior in game theory and economics.[4]

By the same token Lo (1996) used a variant model of knowledge with multiple priors to study strategic choice with ambiguity aversion; see also Chen and Luo (2005) for related discussion on the framework adopted here.

[4] As M. Shubik reminisced, "[von Neumann] felt that it was premature to consider solutions which picked out a single point and he did not like noncooperative equilibrium solutions. In a personal conversation with von Neumann (on the train from New York to Princeton in 1952), I recall suggesting that I thought that Nash's noncooperative equilibrium solution theory might be of considerable value in applications to economics. He indicated that he did not particularly like the Nash solution and that a cooperative theory made more social sense" (Weintraub 1992, pp.155). von Neumann and Morgenstern (1944) clearly expressed their desires and beliefs about the solution concept of a stable set, and asserted that the concept describes "how things are in actual social organizations."

The rest of this paper is organized as follows. Section 2 contains some preliminary notation and definitions. Section 3 discusses stability and rationality in strategic games. Section 4 provides some concluding remarks. To facilitate reading, all the proofs are relegated to the Appendix.

2. Preliminaries

2.1 *General systems and stable sets*

In the spirit of J. von Neumann and O. Morgenstern, Luo (2001) offered a flexible framework for thinking about complex economic and social environments, by means of introducing the notions of a general system and a (general) stable set. Formally, a general system is a pair $\left(X, \left\{\succ^Y\right\}_{Y\subseteq X}\right)$, where X is a nonempty set and $\succ^Y$ is a conditional dominance relation on X. For all $Y \subseteq X$ and for all $x, y \in X$, $y \succ^Y x$ is interpreted to mean that "y dominates x conditionally on Y."

Definition 1. A subset $\mathcal{K} \subseteq X$ is *a (general) stable set* for a general system $\left(X, \left\{\succ^Y\right\}_{Y\subseteq X}\right)$ if it is a vN-M stable set for an abstract game $\left(X, \succ^{\mathcal{K}}\right)$, i.e., a stable set $\mathcal{K}$ satisfies:

1. [internal stability] $\forall x, y \in \mathcal{K}$, $y \not\succ^{\mathcal{K}} x$; and
2. [external stability] $\forall x \in X\backslash\mathcal{K}$, $\exists y \in \mathcal{K}$ such that $y \succ^{\mathcal{K}} x$.

2.2 *A model of knowledge*

Consider a (finite) strategic game:

$$G = \left(N, \{X_i\}_{i\in N}, \{U_i\}_{i\in N}\right),$$

where N is the set of players, X_i is the set of i's (mixed) strategies, and $U_i : \times_{j\in N} X_j \to \Re$ is i's vN-M utility function. A model of knowledge for G is defined as

$$\mathcal{M}\langle G\rangle \equiv \left(\Omega, \{H_i\}_{i\in N}, \{\Psi_i\}_{i\in N}\right).$$

The constituents of $\mathcal{M}\langle G\rangle$ are briefly explained as follows.

- Ω is a finite set of *states* with typical element ω. We refer to a subset of states as an *event*.

- H_i is i's *information partition* with typical element $H_i(\omega)$ that contains ω. The interpretation of $H_i(\omega)$ is that at ω, i knows only that the true state is in the set $H_i(\omega)$.

- $\Psi_i : \Omega \rightrightarrows X_i$ is a correspondence that assigns a (nonempty) subset $\Psi_i(\omega) \subseteq X_i$ to every $\omega \in \Omega$. The set $\Psi_i(\omega)$ is interpreted as i's *choice set* at ω.[5]

Assume, as usual, that the model of knowledge is commonly known, in a meta-sense. In particular, i's payoff function U_i in game G is assumed to be commonly known.

2.3 *Common knowledge*

The idea of common knowledge is by now central to game theory and the economics of uncertainty and information. Roughly speaking, an event is common knowledge if everyone knows it, and everyone knows that everyone knows it, and everyone knows that everyone knows that everyone knows it, and so on *ad infinitum*.

Consider a model of knowledge $\mathcal{M}\langle G\rangle$. For an event E, we take the following standard definitions; see, for instance, Aumann (1995) and Osborne and Rubinstein (1994, Chapter 5).

- $K_iE \equiv \{\omega \in \Omega |\ H_i(\omega) \subseteq E\}$ is the event that i *knows* E.

- $KE \equiv \cap_{i\in N} K_iE$ is the event that *everyone knows* E (i.e., KE is the event that E is mutually known).

- $CKE \equiv KE \cap KKE \cap KKKE \cap \cdots$ is the event that E *is commonly known*.

- E is said to be a *self-evident event* if $KE = E$.

It is well known that $\omega \in CKE$ if, and only if, there is a self-evident event $F \subseteq E$ such that $\omega \in F$. Moreover, CKE is a self-evident event.

[5]For $\omega \in \Omega$, let $\Psi(\omega) \equiv \times_{i\in N}\Psi_i(\omega)$ and $\Psi_{-i}(\omega) \equiv \times_{j\in N\setminus\{i\}}\Psi_j(\omega)$. For $E \subseteq \Omega$, let $\Psi(E) \equiv \cup_{\omega\in E}\Psi(\omega)$ and $\Psi_{-i}(E) \equiv \cup_{\omega\in E}\Psi_{-i}(\omega)$.

3. Stability and rationality in strategic games

3.1. *The general systems of strategic games*
Consider a game $G = \left(N, \{X_i\}_{i \in N}, \{U_i\}_{i \in N}\right)$. Let $X \equiv \times_{j \in N} X_j$ and $X_{-i} \equiv \times_{j \neq i} X_j$. *The general system associated with game* G is defined as a pair $\left(X, \{\succ^Y\}_{Y \subseteq X}\right)$ such that (i) X is the set of strategy profiles, and (ii) for all $x, y \in X$ and for all $Y \subseteq X$

$$y \succ^Y x \text{ iff, for some } i,\ y_i \text{ strictly dominates } x_i \text{ given } Y.$$

That is, $y \succ^Y x$ if, and only if, for some $i \in N$ and all $y_{-i} \in Y_{-i}$, $U_i(y_i, y_{-i}) > U_i(x_i, y_{-i})$, where

$$Y_{-i} \equiv \{x_{-i} \in X_{-i} |\ \exists x_i \in X_i \text{ such that } (x_i, x_{-i}) \in Y\}.$$

It is easy to see that in this associated general system, a stable set must be in Cartesian product form. In particular, the singleton of a strict Nash equilibrium is a stable set (and the singleton of a Nash equilibrium is an internally stable set).

The following example illustrates that, as a natural extension of the vN-M stable set, the idea of the stable set is very useful in the analysis of strategic interaction.
Example 1. Consider the coordination game in Figure 1. For simplicity, only pure strategies are considered.

	L	R
U	$1,1$	$0,0$
D	$0,0$	$1,1$

Fig. 1. A coordination game.

Let $\mathcal{K}^1 = \{U\} \times \{L\}$, let $\mathcal{K}^2 = \{D\} \times \{R\}$, and let $\mathcal{K}^3 = \{U, D\} \times \{L, R\}$. It is easily verified that these three sets are stable sets for the general system associated with this game.

Note that $\mathcal{K}^1 \subset \mathcal{K}^3$ and $\mathcal{K}^2 \subset \mathcal{K}^3$. The sets $\mathcal{K}^1$, $\mathcal{K}^2$, and $\mathcal{K}^3$ cannot be derived from the notion of a vN-M stable set since no two vN-M stable sets can contain one another.

3.2. *Rationality in strategic games*
From a Bayesian viewpoint, when faced with the uncertainty associated with opponents' strategy choices, a player should be able to assign a probability to this uncertainty, i.e.,

a subjective belief about the strategic choice of the other players. Bayesian rationality requires that a player employ only those strategies that are best replies to the beliefs he holds.

For an arbitrary set Z, let $\Delta(Z)$ denote the set of probability distributions on Z. Suppose that within the framework in Subsection 2.2, the potential beliefs that i holds at ω lie in the set $\Delta[\Psi_{-i}(H_i(\omega))]$. For $x_i \in X_i$ and $\mu_i \in \Delta(X_{-i})$, let

$$U_i(x_i, \mu_i) \equiv \int_{X_{-i}} U_i(x_i, x_{-i})\ \mu_i(dx_{-i}).$$

Let $BR_i(\omega)$ denote the set of i's best responses to some of his beliefs in $\Delta[\Psi_{-i}(H_i(\omega))]$. That is,

$$BR_i(\omega) = \{x_i \in X_i |\ x_i \in \arg\max_{y_i \in X_i} U_i(y_i, \mu_i) \text{ for some } \mu_i \in \Delta[\Psi_{-i}(H_i(\omega))]\}.$$

Call i *rational at* ω if $\Psi_i(\omega) = BR_i(\omega)$. In other words, i is rational at a state if i's choice set precisely consists of the strategies that maximize his expected utility with respect to some potential beliefs that i holds at that state.

Let R_i denote the event that i *is rational;* i.e.,

$$R_i = \{\omega \in \Omega |\ \Psi_i(\omega) = BR_i(\omega)\}.$$

Let R denote the event that *everyone is rational;* i.e., $R = \cap_{i \in N} R_i$. Note that, throughout this paper, we do not assume that $\Psi_i(\omega) = \Psi_i(\omega')\ \forall \omega' \in H_i(\omega)$ – i.e. $\Psi_i(\cdot)$ is measurable with respect to H_i; i knows his own choice set. However, if i knows that he is rational, then he also knows his own rational choices (see Lemma 0 in Appendix).

Remark 1. In order to analyze a set-valued solution concept, we define the notion of "rationality" by requiring that the choice set $\Psi_i(\omega)$ consist precisely of all the best responses of player i, with respect to i's own information. It requires not only that every plausible choice of strategy in i's choice set can be "justifiable," but also that any choice of strategy excluded from i's choice set can never be "justified."[6] In the

[6]That is, the rationality of a player requires that: the player's choice set is "closed under rational behavior" in the sense of Basu and Weibull (1991) – i.e., the choice set contains all the best responses of the player; and the player's choice set has the "best response property" in the sense of Pearce (1984) and Bernheim (1984) – i.e., all the strategies in the choice set are a best response of the player.

literature, Samuelson (1992) used a similar notion to discuss the "common knowledge of admissibility." The formulation of rationality in a sense reflects, at an individual level, von Neumann and Morgenstern's (1944) spirit of interpretation of a stable set as an "accepted standard of behavior;" see also Greenberg (1990) for extensive discussions.

3.3. *Foundation of stability*

Consider a strategic game G. Let $\boxed{Y} \equiv \{\omega \in \Omega | \ \Psi(\omega) = Y\}$, i.e., $\boxed{Y}$ is the set of the states where the Cartesian product of the players' choice sets is dictated to Y. An event E is said to be "closed under rational behavior" if, for every player i, $\Psi_i(E)$ contains all the best responses to a possible belief in $\Delta[\Psi_{-i}(E)]$. Apparently, the condition of "closed under rational behavior" can be viewed as a sort of completeness requirement on the discussed model. We are now in a position to state the main results of this section.

Theorem 1. *(i) Let $\omega \in CKR$. Then, $\Psi(\omega)$ is an externally stable set such that there is a commonly-known internally stable set $\mathcal{K} \supseteq \Psi(\omega)$. If, moreover, CKR is closed under rational behavior, then $\Psi(CKR)$ is a stable set. (ii) Let $\boxed{\mathcal{K}}$ be a self-evident event. Then, $\boxed{\mathcal{K}} \subseteq CKR$* whenever *$\mathcal{K}$ is a stable set.*

Remark 2. Following von Neumann and Morgenstern's (1944) interpretation of a stable set, it is fairly natural to assume that the prevailing social norm is mutually known to the public; cf. Lewis (1969). In Shubik's (1982, p.161) words, "the standard of behavior is assumed to be well known to the community." Theorem 1(ii) thus states the important fact that a stable set, as a social norm in a society, describes a state of affairs where rationality is commonly known. A stable set can then be legitimately explained as an "established order of society" or "accepted standard of behavior."

An immediate corollary of Theorem 1 is that CKR determines a subset of (correlated) rationalizable strategy profiles. Formally, we have the following.

Corollary 1. *Every $x \in \Psi(CKR)$ is a rationalizable strategy profile.*

We now provide two examples to illustrate the results in Theorem 1. The first one shows that in Theorem 1(i), neither $\Psi(\omega)$ nor $\Psi(CKR)$ is necessarily stable. The second one shows that the condition given by Theorem 1(ii) is indispensable. For simplicity, only pure strategies are considered.

Example 2. Consider the three-person game in Figure 2. Players 1, 2, and 3 pick the row, column, and matrix, respectively.

A

	L	R
U	$1,1,1$	$0,0,0$
D	$0,0,0$	$1,1,0$

B

	L	R
U	$1,1,0$	$0,0,0$
D	$0,0,0$	$1,1,1$

Fig. 2. A three-person game.

Let $\Omega = \{\omega_1, \omega_2\}$. Let $H_1 = H_2 = \{\{\omega_1\}, \{\omega_2\}\}$, and let $H_3 = \{\Omega\}$. Let $\Psi(\omega_1) = \{U\} \times \{L\} \times \{\mathbf{A}, \mathbf{B}\}$, and let $\Psi(\omega_2) = \{D\} \times \{R\} \times \{\mathbf{A}, \mathbf{B}\}$. Clearly, $CKR = \Omega$.

Firstly, the set $\Psi(\omega_1)$ violates internal stability since, for strategy profiles $(U, L, \mathbf{A})$ and $(U, L, \mathbf{B})$ in $\Psi(\omega_1)$, $(U, L, \mathbf{A}) \succ^{\Psi(\omega_1)} (U, L, \mathbf{B})$ (where player 3's strategy $\mathbf{A}$ strictly dominates $\mathbf{B}$ given that players 1 and 2 respectively play U and L). Secondly, the set $\Psi(CKR)$ violates external stability since, for all $y \in \Psi(CKR)$, $y \not\succ^{\Psi(CKR)} (U, R, \mathbf{A})$, but $(U, R, \mathbf{A}) \notin \Psi(CKR)$.

Example 3. Consider the game in Figure 3.

	L	R
U	$1,1$	$0,0$
D	$0,0$	$1,1$

Fig. 3. A coordination game.

Let $\Omega = \{\omega_1, \omega_2\}$. Let $H_1 = \{\Omega\}$, and let $H_2 = \{\{\omega_1\}, \{\omega_2\}\}$. Let $\Psi(\omega_1) = \{U\} \times \{L\}$, and let $\Psi(\omega_2) = \{U\} \times \{L, R\}$. Clearly, $\Psi(\omega_1)$ is a stable set.

Since $\omega_1 \notin R$, $\omega_1 \notin CKR$. Intuitively, since player 1 is ignorant of $\Psi_2(\omega_1)$ and $\Psi_2(\omega_2)$, the scope of strategic choices at ω_1 cannot be referred to as being self-evident. The result of Theorem 1(ii) cannot be applied in this case.

Now, an interesting question is addressed as follows. Under what conditions can a stable set be obtained? For the case of a "linear" structure state space – i.e. all states can be disposed on a line – we establish a strong result that stability is implied by common knowledge of rationality. Recall that the meet $\mathcal{M}$ of the collection of all players' information partitions is the finest common coarsening of the partitions. Let

$\mathcal{M}^{i,j}$ denote the meet of two players i and j's information partitions. Formally, we have the following.

Theorem 2. *Suppose every player's information partition consists of intervals in a linearly ordered set Ω.*[7] *Let $\omega \in CKR$ satisfying $\mathcal{M}(\omega) = \mathcal{M}^{i,j}(\omega)$ for all i, j. Then, $\Psi(\omega)$ is a stable set.*

Under the "linear" structure assumption, since $\mathcal{M}(\omega) = \mathcal{M}^{1,2}(\omega)$ in two-person games, Theorem 2 yields the following.

Corollary 2. *In two-person games, $\Psi(\omega)$ is a stable set whenever $\omega \in CKR$.*

In games with more than two players, the condition of $\mathcal{M}(\omega) = \mathcal{M}^{i,j}(\omega)$ in Theorem 2 is indispensable. Example 2 demonstrates that if this condition is violated, then $\Psi(\omega)$ may be not stable. The "linear" structure assumption is also essential. Without this assumption, Theorem 2 may fail to be valid even in the case of two-person games. The following example illustrates this point.

Example 4. Consider the two-person game in Figure 4a.

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
x_0	$2,2$	$2,0$	$1,0$	$2,0$	$2,0$	$0,0$	$0,0$
x_1	$0,2$	$0,0$	$2,0$	$0,0$	$0,0$	$0,0$	$0,0$
x_2	$0,0$	$0,0$	$1,1$	$0,2$	$0,0$	$1,1$	$0,0$
x_3	$0,2$	$0,0$	$0,0$	$0,0$	$0,0$	$2,0$	$0,0$
x_4	$0,2$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$	$2,0$
x_5	$0,0$	$0,0$	$0,0$	$0,0$	$0,2$	$1,1$	$1,1$
x_6	$0,0$	$0,2$	$1,1$	$0,0$	$0,0$	$0,0$	$1,1$

Fig. 4a. A two-person game.

Let us consider a model of knowledge for this game as follows (cf. Figure 4b).

- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$
- $H_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$ and $H_2 = \{\{\omega_6, \omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}$

[7] A linearly ordered set is a set with a total order relation on the set. An interval $[\alpha, \beta]$ in Ω is the set of all the states ω in Ω that satisfy $\alpha \leq \omega \leq \beta$; see, e.g., Birkhoff (1967).

$$\bullet \; \Psi(\omega) = \begin{cases} \{x_0, x_1, x_2, x_3\} \times \{y_0, y_1, y_2, y_3\}, \text{ if } \omega = \omega_1 \\ \{x_0, x_1, x_2, x_3\} \times \{y_0, y_3, y_4, y_5\}, \text{ if } \omega = \omega_2 \\ \{x_0, x_3, x_4, x_5\} \times \{y_0, y_3, y_4, y_5\}, \text{ if } \omega = \omega_3 \\ \{x_0, x_3, x_4, x_5\} \times \{y_0, y_1, y_4, y_6\}, \text{ if } \omega = \omega_4 \\ \{x_0, x_1, x_4, x_6\} \times \{y_0, y_1, y_4, y_6\}, \text{ if } \omega = \omega_5 \\ \{x_0, x_1, x_4, x_6\} \times \{y_0, y_1, y_2, y_3\}, \text{ if } \omega = \omega_6 \end{cases}$$

It is easy to verify that $CKR = \Omega$. However, since x_3, for example, is strictly dominated given $\Psi(\omega_1)$, then $\Psi(\omega_1)$ is not a stable set.

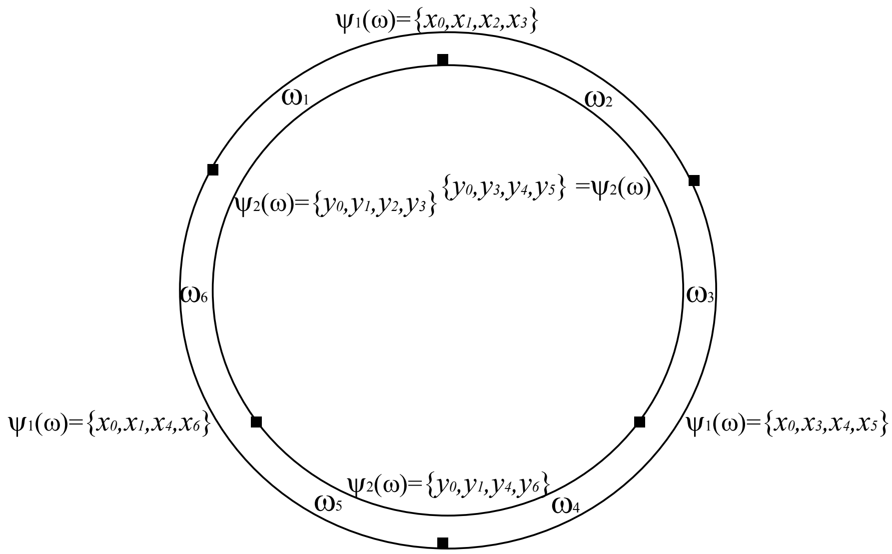

Fig. 4b. An example of a "nonlinear" structure state space.

Next, in the spirit of Aumann and Brandenburger (1995), we provide a sufficient condition for a stable set that whenever the scope of choices is mutually known, rationality alone implies a stable set. Formally, we have the following.

Theorem 3. *Let* $\omega \in R \cap K\boxed{\Psi(\omega)}$. *Then,* $\Psi(\omega)$ *is a stable set.*

If, in particular, every player has a strictly dominant strategy, then rationality alone implies a unique stable set. Formally, we have the following.

Corollary 3. *Let* $\omega \in R$. *If every player has a strictly dominant strategy, then* $\Psi(\omega)$ *is a unique stable set.*

We end this section by establishing the result that any stable set can be attained in terms of CKR.

Theorem 4. *Let $\mathcal{K}$ be a stable set. Then, there exists $\mathcal{M}\langle G\rangle$ such that, for all $\omega \in CKR$, $\Psi(\omega) = \mathcal{K}$.*

An immediate corollary of Theorem 4 is that CKR is possible in every strategic game. Formally, we have the following.

Corollary 4. *There exists $\mathcal{M}\langle G\rangle$ such that $CKR \neq \emptyset$.*

4. Concluding remarks

The purpose of this paper has been to study the epistemic foundation for the notion of stability. The approach in this paper is from a decision-theoretic viewpoint and the analysis is carried out within the standard semantic framework *á la* Aumann (1976). We have established some epistemic conditions for a "stable" pattern of behavior in strategic games, in terms of cognition, information and rationality that lie behind the set-valued solution of the stable set. The rationale behind stability has been found to be abundant by establishing the formal linkage between stability and rationality, so that those committed to employing the notion of stability in their analysis may find it reassuring. As mentioned in the Introduction, von Neumann and Morgenstern (1944) interpreted a stable set as a prevailing norm or an "accepted standard of behavior" in a society, and also pointed out that it describes "how things are in actual social organizations." In this paper we have presented a formal analysis of why, how, and under what conditions the "stable" pattern of behavior can be sustained in the context of strategic interaction. This paper, thus, provides some theoretic rationale for social norms and standards of behavior.

Throughout this paper we have restricted our study to strategic games, but the approach is easily applied to dynamic games; see Luo (2003, Section 4) for a study of the foundation for the notion of stability in extensive games with perfect information. Finally, we would like to point out that in this paper we assume that players are Bayesian rational. Nevertheless, experimental evidence such as the Ellsberg Paradox contradicts some of the tenets in the Savage model of expected utility; see, e.g., Lo's (1996) and Volij's (1996) works on Nash equilibrium under non-expected utility preferences. Extension of this paper to games with general information structures and diverse preferences is also an interesting subject for further research. The interested reader is referred to Chen and Luo (2005) for a more general study on this subject.

Appendix: Proofs

Lemma 0. *If $\omega \in K_i R_i$, then $\Psi_i(\omega) = \Psi_i(\omega')$ for all $\omega' \in H_i(\omega)$.*
Proof. Let $\omega' \in H_i(\omega)$. Then, $H_i(\omega) = H_i(\omega')$. Thus, $\Psi_{-i}(H_i(\omega)) = \Psi_{-i}(H_i(\omega'))$. Therefore, $BR_i(\omega) = BR_i(\omega')$. Since $\omega \in K_i R_i$, $\omega, \omega' \in R_i$. Hence, $\Psi_i(\omega) = \Psi_i(\omega')$.■

To prove Theorem 1, we need the following Lemmas 1-5.
Lemma 1. *Let $Y \subseteq X$ be nonempty and compact. Then, a strategy x_i is strictly dominated given Y if, and only if, it is not a best response to any $\mu_i \in \Delta(Y_{-i})$.*
Proof. Let x_i be strictly dominated given Y. Then, there is $y_i \in X_i$ such that $U_i(y_i, y_{-i}) > U_i(x_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$. Therefore, for all $\mu_i \in \Delta(Y_{-i})$

$$\begin{aligned} U_i(y_i, \mu_i) &= \int_{Y_{-i}} U_i(y_i, y_{-i})\ \mu_i(dy_{-i}) \\ &> \int_{Y_{-i}} U_i(x_i, y_{-i})\ \mu_i(dy_{-i}) \\ &= U_i(x_i, \mu_i). \end{aligned}$$

Hence, x_i is not a best response to any $\mu_i \in \Delta(Y_{-i})$.

Conversely, let x_i be not a best response to any $\mu_i \in \Delta(Y_{-i})$. Consider a zero-sum game $G' \equiv (N', \{X'_j\}_{j\in N'}, \{U'_j\}_{j\in N'})$ such that $N' = \{i, -i\}$, $X'_i = X_i$, $X'_{-i} = \Delta(Y_{-i})$, and $U'_i(y_i, \mu_i) = U_i(y_i, \mu_i) - U_i(x_i, \mu_i)$ for all $y_i \in X'_i$ and $\mu_i \in X'_{-i}$. Clearly, the set X'_j is nonempty, convex, and compact (see, e.g., Pearce's (1984) Lemma 1), and the function $U'_i(y_i, \mu_i)$ is quasi-concave in y_i and continuous. By Glicksberg's (1952) theorem, there exists a Nash equilibrium (y_i^*, μ_i^*) in game G'. Since x_i is not a best response to any $\mu_i \in \Delta(Y_{-i})$, it follows that $\max_{y_i \in X_i} U_i(y_i, \mu_i^*) > U_i(x_i, \mu_i^*)$. Therefore, for any $\mu_i \in \Delta(Y_{-i})$

$$U'_i(y_i^*, \mu_i) \geq U'_i(y_i^*, \mu_i^*) = \max_{y_i \in X'_i} U'_i(y_i, \mu_i^*) > 0.$$

Hence, $U_i(y_i^*, y_{-i}) > U_i(x_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$. That is, x_i is strictly dominated given Y.■
Lemma 2. *Let $\omega \in K_i R$. Then, $\Psi_i(\omega)$ is compact.*
Proof. Since $\omega \in K_i R$, $H_i(\omega) \subseteq R_j$ for all $j \neq i$. By the linearity of $U_j(\cdot, \mu_j)$, it follows that $\Psi_j(\omega)$ has the pure strategy property (i.e., the support of any strategy in $\Psi_j(\omega)$ is a subset of $\Psi_j(\omega)$). Thus, $\Delta[\Psi_{-i}(H_i(\omega))]$ is compact. However, since $\omega \in K_i R$, $\omega \in R_i$. The compactness of $\Psi_i(\omega)$ therefore follows directly from the continuity of U_i and the compactness of $\Delta[\Psi_{-i}(H_i(\omega))]$ (see, e.g., Aliprantis and Border 1999, 16.31 Berge's Maximum Theorem).■

Lemma 3. *Let $E \subseteq R$ be a self-evident event. Then, $\Psi(E)$ is internally stable.*
Proof. Let $x \in \Psi(E)$. Then, there exists $\omega \in E$ such that $x \in \Psi(\omega)$. Since $E \subseteq R$, $\omega \in R_i$ for all $i \in N$. Thus, $x_i \in BR_i(\omega)$ for all $i \in N$. That is, $\forall i \in N$, $\exists \mu_i \in \Delta[\Psi_{-i}(H_i(\omega))]$ such that $x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)$.

Since E is a self-evident event, $H_i(\omega) \subseteq E$. Therefore, $\Psi_{-i}(H_i(\omega)) \subseteq \Psi_{-i}(E)$. Thus, $\forall i \in N$, $\exists \mu_i \in \Delta[\Psi_{-i}(E)]$ such that $x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)$. Since Ω is finite, by Lemma 2 $\Psi(E)$ is compact. By Lemma 1, for all $i \in N$, x_i is not strictly dominated given $\Psi(E)$. Hence, $\Psi(E)$ is an internally stable set.■

Lemma 4. *Every stable set $\mathcal{K}$ has the pure strategy property (i.e., the support of any profile in $\mathcal{K}$ is a subset of $\mathcal{K}$).*
Proof. Suppose Lemma 4 is false. It must be the case that, for some $x \in \mathcal{K}$ and for some i, there is x_i^0 in the support of x_i that is strictly dominated given $\mathcal{K}$. Therefore, there exists $y_i \in X_i$ that strictly dominates x_i given $\mathcal{K}$. Now consider a partial ordered set $(X_i', \succsim)$, such that $X_i' \equiv \{y_i \in X_i |\ y_i \text{ strictly dominates } x_i \text{ given } \mathcal{K}\}$, and for all $x_i', y_i' \in X_i'$,

(a) $y_i' \succ x_i'$ iff y_i' strictly dominates x_i' given $\mathcal{K}$;
(b) $y_i' \sim x_i'$ iff $y_i' = x_i'$.

By the compactness of X_i and the continuity of U_i, it is easily verified that every totally ordered subset of X_i' has an upper bound in X_i'. By making use of Zorn's Lemma (see, e.g., Aliprantis and Border 1999), there is a maximal strategy $\widehat{y}_i \in X_i'$ that strictly dominates x_i given $\mathcal{K}$. Clearly, $\widehat{y}_i$ is not strictly dominated given $\mathcal{K}$. Let $y \equiv (\widehat{y}_i, x_{-i})$. Since $x \in \mathcal{K}$, it follows that $x, y \in \mathcal{K}$ and $y \succ^{\mathcal{K}} x$, contradicting the internal stability of $\mathcal{K}$.■

Lemma 5. *Every stable set $\mathcal{K}$ is nonempty and compact.*
Proof. Note that $U_i(y_i, y_{-i}) > U_i(x_i, y_{-i})\ \forall y_{-i} \in \mathcal{K}_{-i}$ if, and only if, $U_i(y_i, \mu_i) > U_i(x_i, \mu_i)$ $\forall \mu_i \in \Delta(\mathcal{K}_{-i})$. By Lemma 4, it is easy to see that $\Delta(\mathcal{K}_{-i})$ is compact. By the proof of Lemma 1, $x \in \mathcal{K}$ if, and only if, there exists $\mu_i \in \Delta(\mathcal{K}_{-i})$ such that $x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)$. Let $BR_i[\Delta(\mathcal{K}_{-i})]$ denote

$$\{x_i \in X_i |\ \exists \mu_i \in \Delta(\mathcal{K}_{-i}) \text{ s.t. } U_i(x_i, \mu_i) \geq U_i(y_i, \mu_i) \quad \forall y_i \in X_i\}.$$

Thus, $\mathcal{K} = \times_{i \in N} BR_i[\Delta(\mathcal{K}_{-i})]$. But, by the continuity of U_i, it follows that $BR_i[\Delta(\mathcal{K}_{-i})]$ is a compact set. Hence, $\mathcal{K}$ is compact. The nonemptiness follows directly from the external stability of $\mathcal{K}$.■

We now turn to the proof of Theorem 1.
Proof of Theorem 1. The proof of (i) is split into two parts.

Part 1. Let $x \in X \backslash \Psi(\omega)$. Then, $x_i \notin BR_i(\omega)$ for some i. By Lemma 2, $\Psi(H_i(\omega))$ is nonempty and compact. By Lemma 1, there exists $y_i \in X_i$ that strictly dominates x_i given $\Psi(H_i(\omega))$. Therefore, there is $y_i \in X_i$ that strictly dominates x_i given $\Psi(\omega)$. Consider a partial ordered set $(X_i', \succcurlyeq)$, such that $X_i' \equiv \{y_i \in X_i |\ y_i$ strictly dominates x_i given $\Psi(\omega)\}$, and for all $x_i', y_i' \in X_i'$,

(a) $y_i' \succ x_i'$ iff y_i' strictly dominates x_i' given $\Psi(\omega)$;

(b) $y_i' \sim x_i'$ iff $y_i' = x_i'$.

By the compactness of X_i and the continuity of U_i, it is easily verified that every totally ordered subset of X_i' has an upper bound in X_i'. By making use of Zorn's Lemma, there is a maximal strategy $\widehat{y}_i \in X_i'$ that strictly dominates x_i given $\Psi(\omega)$. By Lemmas 1 and 2, it follows that $\widehat{y}_i \in BR_i(\omega)$. Since $\omega \in R_i$, $\widehat{y}_i \in \Psi_i(\omega)$. Let $y \in X$ be such that, for all i,

$$y_i = \begin{cases} \widehat{y}_i, \text{ if } x_i \notin BR_i(\omega) \\ x_i, \text{ if } x_i \in BR_i(\omega) \end{cases}.$$

Clearly, $y \in \Psi(\omega)$ and $y \succ^{\Psi(\omega)} x$. Therefore, $\Psi(\omega)$ is an externally stable set.

Part 2. Let $\mathcal{K} = \Psi(CKR)$. Clearly, $\mathcal{K} \supseteq \Psi(\omega)$. Since CKR is a self-evident event, by Lemma 3, $\mathcal{K}$ is a commonly-known internally stable set. Now suppose CKR is closed under rational behavior. By the proof of Lemma 2, $\Psi(CKR)$ is compact. Similar to the proof of Part 1, $\Psi(CKR)$ is an externally stable set. Consequently, $\Psi(CKR)$ is a stable set if CKR is closed under rational behavior.

(ii) Let $\omega \in \boxed{\mathcal{K}}$. Let $\mathcal{M}(\omega)$ denote the member of the meet of the players' information partitions that contains ω (see Aumann 1976). Since $\boxed{\mathcal{K}}$ is a self-evident event, $\mathcal{M}(\omega) \subseteq \boxed{\mathcal{K}}$. Therefore, $\Psi(\omega') = \mathcal{K}$ for all $\omega' \in \mathcal{M}(\omega)$. Since for all i and for all $\omega' \in \mathcal{M}(\omega)$, $H_i(\omega') \subseteq \mathcal{M}(\omega)$, $\Psi_{-i}(H_i(\omega')) = \mathcal{K}_{-i}$.

By Lemmas 1 and 5, it therefore follows that: $\left[\mathcal{K} \text{ is a stable set}\right] \Rightarrow \left[x \in X \backslash \mathcal{K}\right.$ if and only if $y \succ^{\mathcal{K}} x$ for some $\left.y \in \mathcal{K}\right] \Rightarrow \left[x \in X \backslash \mathcal{K}\right.$ if and only if, for some i and for all $\mu_i \in \Delta(\mathcal{K}_{-i})$, $\left.x_i \notin \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)\right] \Rightarrow \left[x \in \mathcal{K}\right.$ if and only if, for all i, $\exists$ $\mu_i \in \Delta(\mathcal{K}_{-i})$ such that $\left.x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)\right] \Rightarrow \left[\Psi_i(\omega') = BR_i(\omega')\right.$ for all i and for all $\left.\omega' \in \mathcal{M}(\omega)\right] \Rightarrow \left[\mathcal{M}(\omega) \subseteq R\right]$. Hence, $\omega \in CKR$.■

Proof of Corollary 1. Since $x \in \Psi(CKR)$, $x \in \Psi(\omega)$ for some $\omega \in CKR$. Therefore, $\forall i \in N$, $\exists \mu_i \in \Delta[\Psi_{-i}(H_i(\omega))]$ such that $x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)$. However, since CKR is self-evident, $H_i(\omega) \subseteq CKR$. Therefore, $\forall i \in N$, $\exists \mu_i \in \Delta[\Psi_{-i}(CKR)]$ such that $x_i \in \operatorname{argmax}_{y_i \in X_i} U_i(y_i, \mu_i)$. Hence, $x \in \Psi(CKR)$ is a rationalizable strategy profile.■

To prove Theorem 2, we need the following Lemma 6. According to i's information partition, let $\mathcal{M}(\omega)$ consist of m_i (discrete) intervals $\left[\alpha_i^1, \beta_i^1\right], \left[\alpha_i^2, \beta_i^2\right], \ldots,$ and $[\alpha_i^{m_i}, \beta_i^{m_i}]$ and let $\Psi_i^1, \Psi_i^2, \ldots,$ and $\Psi_i^{m_i}$ be i's choice sets over these intervals, respectively (cf. Figure 5 below).

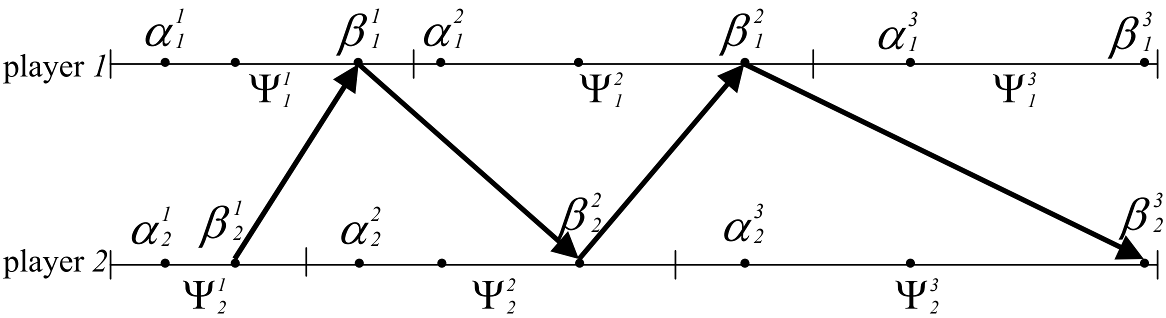

Fig. 5. An example of $\mathcal{M}(\omega)$ under the "linear" structure assumption.

Lemma 6. $\Psi_i(\cdot)$ *is measurable with respect to* $\mathcal{M}(\omega)$.

Proof of Lemma 6. (i) For $i = 1, 2, \ldots, n$, let $\beta_i^{\imath(0)} \equiv \beta_i^1$. Define

$$\beta_i^{\imath(1)} \equiv \begin{cases} \beta_i^{\imath(0)+1}, & \text{if } \beta_i^{\imath(0)} < \min_{j \neq i} \beta_j^{\imath(0)}; \\ \beta_i^{\imath(0)}, & \text{otherwise.} \end{cases}$$

Clearly, $\Psi_{-i}\left(\beta_i^{\imath(0)}\right) = \Psi_{-i}\left(\beta_i^1\right) \subseteq \Psi_{-i}\left(\beta_i^2\right) = \Psi_{-i}\left(\beta_i^{\imath(1)}\right)$ whence $\beta_i^1 < \min_{j \neq i} \beta_j^1$; $\beta_i^{\imath(0)} = \beta_i^1 = \beta_i^{\imath(1)}$ whence $\beta_i^1 \geq \min_{j \neq i} \beta_j^1$. Therefore, $\Psi_{-i}\left(\beta_i^{\imath(0)}\right) \subseteq \Psi_{-i}\left(\beta_i^{\imath(1)}\right)$ and, hence, $\Psi_i^{\imath(0)} \subseteq \Psi_i^{\imath(1)}$. For all $k \geq 2$, define recursively

$$\beta_i^{\imath(k)} \equiv \begin{cases} \beta_i^{\imath(k-1)+1}, & \text{if } \beta_i^{\imath(k-1)} < \min_{j \neq i} \beta_j^{\imath(k-1)}; \\ \beta_i^{\imath(k-1)}, & \text{otherwise.} \end{cases}$$

Similarly, it is easy to see that: $\Psi_{-i}\left(\beta_i^{\imath(k-1)}\right) \subseteq \Psi_{-i}\left(\beta_i^{\imath(k-1)+1}\right)$ if $\beta_i^{\imath(k-1)} < \min_{j \neq i} \beta_j^{\imath(k-1)}$; $\Psi_{-i}\left(\beta_i^{\imath(k-1)}\right) = \Psi_{-i}\left(\beta_i^{\imath(k)}\right)$ if $\beta_i^{\imath(k-1)} \geq \min_{j \neq i} \beta_j^{\imath(k-1)}$. Therefore, $\Psi_i^{\imath(k)} \subseteq \Psi_i^{\imath(k+1)}$ for all $k \geq 0$. Since Ω is finite, it follows, through this procedure, that $\Psi_i^1 \subseteq \Psi_i^2 \subseteq \ldots \subseteq \Psi_i^{m_i}$.

(ii) For $i = 1, 2, \ldots, n$, let $\alpha_i^{\imath(0)} = \alpha_i^{m_i}$. For all $k \geq 1$, define recursively

$$\alpha_i^{\imath(k)} \equiv \begin{cases} \alpha_i^{\imath(k-1)-1}, & \text{if } \alpha_i^{\imath(k-1)} > \max_{j \neq i} \alpha_j^{\imath(k-1)}; \\ \alpha_i^{\imath(k-1)}, & \text{otherwise.} \end{cases}$$

It is easily verified that: $\Psi_{-i}\left(\alpha_i^{\imath(k-1)}\right) \subseteq \Psi_{-i}\left(\alpha_i^{\imath(k-1)-1}\right)$ if $\alpha_i^{\imath(k-1)} > \max_{j\neq i} \alpha_j^{\imath(k-1)}$; $\Psi_{-i}\left(\alpha_i^{\imath(k-1)}\right) = \Psi_{-i}\left(\alpha_i^{\imath(k)}\right)$ if $\alpha_i^{\imath(k-1)} \leq \max_{j\neq i} \alpha_j^{\imath(k-1)}$. Therefore, $\Psi_i^{\imath(k)} \subseteq \Psi_i^{\imath(k+1)}$ for all $k \geq 0$. Since Ω is finite, it follows, through this procedure, that $\Psi_i^{m_i} \subseteq \Psi_i^{m_i-1} \subseteq \ldots \subseteq \Psi_i^1$.

By (i) and (ii), $\Psi_i^1 = \Psi_i^2 = \ldots = \Psi_i^{m_i}$.■

We now turn to the proof of Theorem 2.

Proof of Theorem 2. Since $\mathcal{M}(\omega)$ is a self-evident event, by Lemma 3, $\Psi(\mathcal{M}(\omega))$ is an internally stable set. By Lemma 6, $\Psi(\omega) = \Psi(\mathcal{M}(\omega))$. Therefore, $\Psi(\omega)$ is internally stable. However, by Theorem 1, $\Psi(\omega)$ is also externally stable. Hence, $\Psi(\omega)$ is a stable set.■

Proof of Corollary 2. Since $\mathcal{M}(\omega) = \mathcal{M}^{1,2}(\omega)$ in two-person games, the result follows directly from Theorem 2.■

Proof of Theorem 3. Since $\omega \in K\boxed{\Psi(\omega)}$, $\omega \in K_i\boxed{\Psi(\omega)}$ for all i. Thus, $H_i(\omega) \subseteq \boxed{\Psi(\omega)}$. Therefore, $\Psi(\omega') = \Psi(\omega)$ for all $\omega' \in H_i(\omega)$. Hence, $\Psi_{-i}[H_i(\omega)] = \Psi_{-i}(\omega)$. However, since $\omega \in R$, $\omega \in R_i$. Thus, $\Psi_i(\omega) = BR_i(\omega)$. That is, $x_i \in \Psi_i(\omega)$ if and only if $\exists \mu_i \in \Delta[\Psi_{-i}(\omega)]$ such that $x_i \in \text{argmax}_{y_i\in X_i} U_i(y_i, \mu_i)$.

By the linearity of $U_j(\cdot, \mu_j)$, $\Psi_j(\omega)$ has the pure strategy property and, thus, $\Delta(\Psi_{-i}(\omega))$ is compact. Therefore, for all i, $\Psi_i(\omega)$ is compact. By Lemma 1, it follows that $x \in X\backslash\Psi(\omega)$ if and only if, for some i, there exists $y_i \in X_i$ such that y_i strictly dominates x_i given $\Psi(\omega)$. By making use of Zorn's Lemma, it is easily verified that $x \in X\backslash\Psi(\omega)$ if and only if, for some i, there exists $y_i \in \Psi_i(\omega)$ such that y_i strictly dominates x_i given $\Psi(\omega)$. Hence, $x \in X\backslash\Psi(\omega)$ if, and only if, there exists $y \in \Psi(\omega)$ such that $y \succ^{\Psi(\omega)} x$.■

Proof of Corollary 3. Let x_i^* be i's strictly dominant strategy. Since $\omega \in R$, $\Psi_i(\omega) = \{x_i^*\}$ for all i. Therefore, $\Psi(\omega) = \{x^*\}$, which is the unique stable set.■

Proof of Theorem 4. We can define a model of knowledge $\mathcal{M}\langle G\rangle$ by specifying that Ω consists of a single state ω and $\Psi(\omega) = \mathcal{K}$. By the proof of Lemma 5, $\mathcal{K} = \times_{i\in N} BR_i[\Delta(\mathcal{K}_{-i})]$. Therefore, for all i, $x_i \in \Psi_i(\omega)$ if and only if $\exists \mu_i \in \Delta[\Psi_{-i}(\omega)]$ such that $x_i \in \text{argmax}_{y_i\in X_i} U_i(y_i, \mu_i)$. Thus, $\omega \in R$. Hence, $CKR = \{\omega\}$. That is, $\Psi(\omega) = \mathcal{K}$ for all $\omega \in CKR$.■

Proof of Corollary 4. Note that the set of profiles surviving iterated elimination of strictly dominated strategies is a stable set. By Theorem 4, there exists $\mathcal{M}\langle G\rangle$ such that $CKR \neq \emptyset$.■

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