ON THE DEFINABLE IDEAL GENERATED BY THE PLUS CUPPING C.E. DEGREES

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Abstract. In this paper we will prove that the plus cupping degrees generate a definable ideal on c.e. degrees different from other ones known so far, thus answer a question asked by A. Li and Yang.

1. Introduction

The study of definable structures has been an interesting topic in degree theory for years. The following theorem is arguably the most elegant result of this kind in the theory of computably enumerable degrees.

Theorem 1.1 (Ambos-Spies, Jockusch, Shore and Soare [1]). *Let* $\mathbf{M}$ *denote the collection of cappable c.e. degrees and* $\overline{\mathbf{M}} = \mathbf{R} - \mathbf{M}$ *where* $\mathbf{R}$ *is the collection of c.e. degrees, then* $\mathbf{M}$ *forms a definable prime ideal and* $\overline{\mathbf{M}}$ *forms a strong ultra filter in* $\mathbf{R}$.

In [1], the equivalence of $\overline{\mathbf{M}}$ and several other subsets of $\mathbf{R}$ is also established, e.g., $\overline{\mathbf{M}} = \mathbf{PS} = \mathbf{LC}$ where $\mathbf{PS}$ is the collection of promptly simple degrees and $\mathbf{LC}$ is of low cuppable degrees. Despite of this important finding, $\mathbf{M}$ and $\mathbf{NCup}$ (the collection of noncuppable degrees) remained the only known definable ideals in $\mathbf{R}$ for quite a long time. This situation led to the following question.

Question 1.2 (Shore [9]). *Are there any definable ideals in* $\mathbf{R}$ *other that* $\mathbf{M}$ *and* $\mathbf{NCup}$.

Later Nies proved the following powerful theorem which suggests an effective way of finding definable ideals.

Theorem 1.3 (Nies [8]). *Given* $\mathbf{X}$ *a definable subset of* $\mathbf{R}$, *the ideal generated by* $\mathbf{X}$ *(denoted by* $[\mathbf{X}]$*) is also definable in* $\mathbf{R}$.

Yu and Yang found the other definable ideals known so far using the theorem above.

Theorem 1.4 (Yu and Yang [11]). *Let* $\mathbf{NB}$ *denote the collection of nonbounding c.e. degrees, then* $[\mathbf{NB} \cup \mathbf{NCup}]$ *is a proper subideal of* $\mathbf{M}$. *Furthermore,* $[\mathbf{NB}] \cap \mathbf{NCup}$, $[\mathbf{NB}]$, $\mathbf{NCup}$ *and* $[\mathbf{NB} \cup \mathbf{NCup}]$ *are different from each other.*

Another candidate of definable ideals is relating to $\mathbf{PC}$, i.e., the collection of plus cupping degrees. Let us recall the definition of plus cupping degrees.

Definition 1.5 (Harrington [3], Fejer and Soare [2]). *A c.e. degree* $\mathbf{a}$ *is* plus-cupping *if and only if for every nonrecursive* $\mathbf{b} \leq_{\mathrm{T}} \mathbf{a}$, *there is an incomplete c.e. degree* $\mathbf{c}$ *such that* $\mathbf{b} \cup \mathbf{c} = \mathbf{0}'$.

As remarked by D. Li and A. Li [4], the typical plus cupping constructions resemble those of nonbounding degrees to some extent. However, these two notions are different.

Theorem 1.6 (D. Li and A. Li [4])**.** $\mathbf{PC} - \mathbf{NB} \neq \emptyset$.

In addition A. Li and Y. Zhao proved the following.

Theorem 1.7 (Li and Zhao [7])**.** *Plus cupping degrees do not form an ideal.*

Based on these facts, A. Li and Yang asked the following question.

Question 1.8 (A. Li and Yang [6])**.** *Is* $[\mathbf{PC}]$ *different from* $[\mathbf{NB}]$*?*

In this paper, we will answer this question affirmatively. Actually we will prove a stronger result that $[\mathbf{PC}]$ is a proper subideal of $\mathbf{M}$ not contained by $[\mathbf{NB} \cup \mathbf{NCup}]$. For this sake, in section 2 we will prove that $\mathbf{NCup}$ is not a subset of $[\mathbf{PC}]$, hence $[\mathbf{PC}]$ is a proper subideal of $\mathbf{M}$; while in section 3, we will prove that $[\mathbf{PC}]$ is not contained by $[\mathbf{NB} \cup \mathbf{NCup}]$.

For notions and conventions we follow Soare [10]. Sets and functionals defined in the proofs should be considered computably enumerable unless additionally indicated.

2. $\mathbf{NCup} \not\subseteq [\mathbf{PC}]$

Theorem 2.1. *There is a noncuppable c.e. degree* $\mathbf{a} \notin [\mathbf{PC}]$*.*

We prove Theorem 2.1 by constructing a c.e. set A such that $\mathbf{deg}(A) \in \mathbf{NCup}$ and $\mathbf{deg}(A) \notin [\mathbf{PC}]$.

To make A noncuppable, fix a computable enumeration $(\Phi_e, W_e)_{e\in\omega}$ of c.e. functionals and c.e. sets, we build an additional c.e. set D such that for all e

$$\mathcal{M}_e : D = \Phi_e(A, W_e) \Rightarrow K \leq_{\mathrm{T}} W_e$$

To make $\mathbf{deg}(A) \notin [\mathbf{PC}]$, fix a computable coding of $\omega^{<\omega}$. For e let $\|e\|$ denote the unique c such that e codes an element, say z, of ω^{c+1}; and let e_i denote the i-th element of z. Fix $(\Psi_{e_{\|e\|}}, B_{e_0}, B_{e_1}, \dots B_{e_{\|e\|-1}})_{e\in\omega}$, we satisfy the followings requirements for all e

$$\mathcal{P}_e : A = \Psi_{e_{\|e\|}}(B_e) \Rightarrow (\exists i < \|e\|)(B_{e_i} \text{ is not plus cupping})$$

where B_e is the abbreviation of $(B_{e_0}, B_{e_1}, \dots, B_{e_{\|e\|-1}})$.

We arrange the construction on a tree of strategies growing upward. Every finite path of the tree is an $\mathcal{X}$-strategy for some requirement $\mathcal{X}$.

2.1. **$\mathcal{M}$-strategies.** Suppose α is an $\mathcal{M}_e$-strategy. We define l^α the length of agreement between D and $\Phi(A, W)$ and α-expansionary stages as usual.

α has two outcomes ∞ (if there are infinitely many α-expansionary stages) and 0 (if there are at most finitely many).

If there are infinitely many expansionary stages, α builds a p.r. functional Θ^α such that for all k

$$\mathcal{N}_k^\alpha : D = \Phi_e(A, W_e) \Rightarrow K(k) = \Theta^\alpha(W_e; k).$$

To satisfy $\mathcal{N}_k^\alpha$ and define $\Theta^\alpha(W_e; k)$, we arrange $\mathcal{N}_k^\alpha$-strategies above $\alpha\hat{\ }\infty$. From now on in this subsection, we occasionally omit α from superscripts.

Suppose $\beta \supseteq \alpha\hat{}\infty$ is an $\mathcal{N}^{\alpha}_{k}$-strategy. At the beginning, β picks a *flip point* $d^{\beta}(k)$ of k and keeps it from entering D. We may write d for $d^{\beta}(k)$.

If the computation $\Phi_e(A, W_e; d)$ changes infinitely often, β will have $\perp$ as outcome indicating that $\Phi_e(A, W_e; d)$ diverges. In this case, we arrange no more $\mathcal{N}^{\alpha}$-strategies above $\beta\hat{}\perp$ since $D \neq \Phi_e(A, W_e)$.

Otherwise β has $\top$ as outcome and defines $\Theta(W_e; k) = K(k)$ with $\theta(k) > \phi_e(d)$. In addition, β expects that $A \upharpoonright \phi_e(d)$ changes no longer.

If k is enumerated in K later, β enumerates d in D, then either β establishes a disagreement between D and $\Phi_e(A, W_e)$, or $W_e \upharpoonright \phi_e(d)$ eventually changes and β can safely change the definition of $\Theta(W_e; k)$ to 1.

2.2. **$\mathcal{P}$-strategies.** Suppose τ is an $\mathcal{P}_e$-strategy. We define l^{τ} the length of agreement between D and $\Phi(A, W)$ and τ-expansionary stages as usual.

τ has two outcomes ∞ (if there are infinitely many τ-expansionary stages) and 0 (if there are at most finitely many). If there are infinitely many expansionary stages, τ builds $\|e\|$ many c.e. sets $(C^{\tau}_0, C^{\tau}_1, \ldots, C^{\tau}_{\|e\|-1})$ such that $C^{\tau}_i \leq_{\mathrm{T}} B_{e_i}$ for $i < \|e\|$, for one of i

$$\mathcal{Q}^{\tau}_{i,j} : C^{\tau}_i \neq \overline{W}_j,$$

and for $(i, j) \in \|e\| \times \omega$

$$\mathcal{R}^{\tau}_{i,j} : D = \Phi_j(C^{\tau}_i, W_j) \Rightarrow K \leq_{\mathrm{T}} W_j.$$

To satisfy $\mathcal{Q}^{\tau}_{i,j}$ and $\mathcal{R}^{\tau}_{i,j}$, we arrange $\mathcal{Q}^{\tau}_{i,j}$- and $\mathcal{R}^{\tau}_{i,j}$-strategies above $\tau\hat{}\infty$. From now on in this subsection, we may occasionally omit τ from superscripts.

Suppose $\alpha \supseteq \tau\hat{}\infty$ is an $\mathcal{R}^{\tau}_{i,j}$-strategies, α acts in the same way as an $\mathcal{M}_e$-strategy described in the previous subsection. α has two outcome ∞ (indicating there are infinitely many α-expansionary stages) and 0 (indicating there are at most finitely many), and builds a p.r. functional Θ^{α} such that for all k

$$\mathcal{S}^{\alpha}_{k} : K = \Phi_j(C_i, W_j) \Rightarrow K_0(k) = \Theta^{\alpha}(W_j; k).$$

To satisfy $\mathcal{S}^{\alpha}_{k}$, we arrange $\mathcal{S}^{\alpha}_{k}$-strategies above $\alpha\hat{}\infty$. $\mathcal{S}^{\alpha}_{k}$-strategies act in the same way as $\mathcal{N}$-strategies above $\mathcal{M}$-strategies.

To make one of C_i's non-computable, we do not satisfy every $\mathcal{Q}_{i,j}$. Actually, we satisfy combinations of $\mathcal{Q}$'s

$$\mathcal{Q}^{\tau}_{n} : \bigvee_{i<\|e\|} C_i \neq \overline{W}_{n_i}, \text{ where } \|n\| = \|e\| - 1.$$

We will arrange $\mathcal{Q}_n$-strategies on the tree of strategies so that we can make

$$(\forall j)(C_i \neq \overline{W}_j)$$

for at least one $i < \|e\|$, along every infinite path of the tree (suppose that the assumption $A = \Psi_e(B_e)$ holds).

To additionally make $C_i \leq_{\mathrm{T}} B_{e_i}$ we use *permitting at τ-expansionary stages.* τ will build a local version of effective enumeration of B, i.e., $B^{\tau}[s] = B[s_0]$ where $s_0 \leq s$ is the latest stage when τ is accessible and $\{B[s] | s \in \omega\}$ is some standard enumeration. The computation $\Psi_e(B_e)$ is also localized, i.e., (for τ and its substrategies) it could change only if τ is accessible. From now on we may occasionally identify these localizations with the standard ones.

For $\sigma \supseteq \tau\hat{}\infty$ a $\mathcal{Q}_n$-strategy, at the beginning σ picks an *agitator* a so that $l^{\tau} > a$, and keeps a from entering A. If $B_e \upharpoonright \psi_e(a)$ changes infinitely often, σ has $\perp$ as its outcome indicating that $\Psi_e(B_e; a)$ diverges. Otherwise, σ will eventually fix a *witness* x. If x is never enumerated in W_{n_i} for some $i < \|e\|$, σ has w_i as its outcome. In this case $\mathcal{Q}_{i,n_i}$ is satisfied since $\overline{W}_{n_i} - C_i$ is not empty.

Otherwise at some stage $x \in W_{n_i}$ for all $i < \|e\|$, σ enumerates a in A. If the assumption $A = \Phi_e(B_e)$ is true, then B_{e_i} changes for some i before $A(a) = \Psi_e(B_e; a)$ is established again. We enumerate x in C_i for the least such i. In this case, σ has c_i as its outcome, $\mathcal{Q}_{i,n_i}$ is satisfied since $C_i - \overline{W}_{n_i}$ is not empty.

We will not arrange any $\mathcal{Q}_{n'}$-strategies above $\sigma\hat{}\perp$. While above $\sigma\hat{}w_i$ or $\sigma\hat{}c_i$, we will not arrange other $\mathcal{Q}_{n'}$-strategies with $n'_i = n_i$.

2.3. **Coordinating different strategies.** Since $\mathcal{Q}$-strategies may enumerate their agitators in A while $\mathcal{N}_{e,k}$-strategies expect that $A \upharpoonright \phi(d(k))$'s will never change after $\Theta(W; k)$'s are defined, conflicts arise.

The technique to solve these conflicts is originally developed by Li, Slaman and Yang [5] and then applied by Yu and Yang [11]. However we will give a slightly different formulation hope that the behaviors of flip points could be made clearer.

On the one hand, whenever a $\mathcal{N}$-strategy β defines $\Theta(W; k)$, strategies properly dominated by β are initialized.

On the other hand, the situation is a little more complex. Suppose τ is some $\mathcal{P}_{e'}$-strategy, σ is a $\mathcal{Q}^\tau$-strategy and α is some $\mathcal{M}_e$-strategy dominating σ. At stage s, $\Theta(W; k)[s]$ becomes defined by some $\mathcal{N}^\alpha_k$-strategy β at $s_0 \le s$ and σ intends to enumerate a^σ in A.

If β dominates σ or a^σ is chosen after s_0 then we can easily make $a^\sigma > \phi_e(d^\beta(k))$. Otherwise, in general σ enumerates $d^\beta(k)$ in D to force $W_e \upharpoonright \phi_e(d^\beta(k))$ change. If $W_e \upharpoonright \phi_e(d^\beta(k))$ never changes, then the disagreement $\Phi_e(A, W_e; d^\beta(k)) \neq D(d^\beta(k))$ is established; otherwise $\Theta^\alpha(W_e; k)$ diverges eventually and the enumeration of a^σ in A will not harm the intention to make $\Theta^\alpha(W_e; k)) = K(k)$.

But there is a special case. Assume there is another $\mathcal{N}^\alpha_k$-strategy $\gamma \subset \gamma\hat{}\top \subseteq \sigma$. If the above happens infinitely often (by infinitely many $\mathcal{Q}$-strategies above $\gamma\hat{}\top$) for α and k, then $\Theta^\alpha(W_e; k)$ diverges even though γ might be a *true* strategy for $\mathcal{N}^\alpha$ and $\top$ might be the *true* outcome of γ.

To overcome this difficulty, first, we will allow σ above $\gamma\hat{}\top$ to change $A \upharpoonright \phi_e(d^\beta(k))$ freely. Second, we will make $a^\sigma > \phi_e(d^\gamma(k))$. Assume this is achieved. If later some other strategy wants to make $\Theta^\alpha(W_e k)$ diverges, it can enumerate $d^\gamma(k)$ (instead of $d^\beta(k)$) in D.

To keep track of $d^\gamma(k)$ we introduce a new parameter $d^\alpha(k)$, called *the official flip point of* α *and* k, and assign it to α. We then call $d^\gamma(k)$ *the personal flip point of* γ. Whenever $\gamma\hat{}\top$ is accessible, the official point is defined to be the personal flip point of γ.

Furthermore if later W_e changes below $\theta^\alpha(k)$ but not $\phi_e(d^\alpha(k))$ then $\theta^\alpha(k)$ will not be changed. This guarantees that $\Theta^\alpha(W_e; k)$ converges.

However $\psi_{e'}(a^\sigma)$ might become $\ge x^\sigma$ when σ is waiting for the link (α, σ) to be traveled. If this happens infinitely often, $\Theta^\alpha(W_e; k)$ could diverge. But note that then $\Psi_{e'}(B_{e'}; a^\sigma)$ also diverges. Moreover, this could not happen if τ is covered by the link (α, σ) (or in other words if $\alpha \subset \tau$), since then (for τ and σ) the local computation of $\Psi_{e'}(B_{e'}; a^\sigma)$ will not change until τ is accessible again.

Hence we could just arrange a backup strategy α' for α above $\sigma\hat{}\perp$. Furthermore we will only arrange $\mathcal{N}^{\alpha'}$-strategies but no $\mathcal{N}^\alpha$-strategies above $\sigma\hat{}\perp$. We also backup those $\mathcal{P}$-strategies between τ and σ to guarantee that eventually this backup operation will not happen for $\mathcal{M}_e$ on every infinite path of the tree of strategies.

Now we formally describe procedures for $\mathcal{N}$- and $\mathcal{Q}$-strategies.

Let $s_0 = \max\{s' < s : d^\beta[s] = d^\beta[s']$ and β is accessible at $s'\}$ and $s_1 = \max\{s' < s : \theta(k)[s']$ is defined $\}$.

Procedure 2.2. *Suppose that β is an $\mathcal{N}_{e,k}$-strategy and $\alpha = top(\beta)$. At stage s, β cancels $d^\beta(k)$ if $d^\beta(k) \in D$ and then acts step by step as followings.*

(1) *If $d^\beta(k)$ is undefined, define it to be fresh.*
(2) *If $d^\beta(k) > l$, do nothing and stop.*
(3) *If $d^\beta(k) \leq l$, take the following actions*
    (a) *If s_0 is defined and the computation $\Phi_e(A, W_e; d^\beta(k))[s]$ is different from that at s_0, let $\perp$ be the outcome.*
    (b) *From now on assume (a) fails. If $\Theta(W_e; k)$ diverges, define $\Theta(W_e; k) = K(k)$ with $\theta(k) = \theta(k)[s_1]$ if $d^\alpha(k)$ is defined, or $\theta(k)$ fresh otherwise.*
    (c) *Let $d^\alpha(k) = d^\beta(k)$ if either $d^\alpha(k)$ is undefined or $d^\beta(k) < d^\alpha(k)$.*
    (d) *If $\Theta(W_e; k) \neq K(k)$ then enumerate $d^\alpha(k)$ in D, cancel $d^\beta(k)$ and stop; otherwise let $\top$ be the outcome.*

Note that the conflicts between $\mathcal{S}$-strategies and $\mathcal{Q}$-strategies are similar. Hence we also apply the above procedure for $\mathcal{S}$-strategies.

For $\mathcal{Q}$-strategies, we follow some settings in Yu and Yang [11]. A $\mathcal{Q}_{e,n}$-strategy σ clears θ's of $\mathcal{M}$- and $\mathcal{R}$-strategies α's with $\alpha\hat{}\infty \subseteq \sigma$ in descending order (with respect to $\subset$). To prevent Θ's from being defined on new arguments, σ will setup a link (α, σ) as it enumerates some official flip point in D; the links will be cancelled at next α-expansionary stage and the control will be passed immediately to σ. Moreover, to have the enumeration of its witness in some C promptly permitted by B^τ (where $\tau = top(\sigma)$), σ will setup a link (τ, σ) as it enumerates its agitator in A; the link will be cancelled at next τ-expansionary stage and the control will be passed immediately to σ.

We assign $2\|n\| + 3$ states $\{c_0, c_1, \ldots, c_{\|n\|}, c, w\perp, w_0, w_1, \ldots, w_{\|n\|}\}$ and a parameter $state(\sigma)$ for σ.

Let $s_0 = \max\{s' < s : a^\sigma[s] = a^\sigma[s_0]$ and σ is accessible at $s'\}$.

Procedure 2.3. *At the beginning of stage s, σ picks a fresh agitator a^σ if a^σ is undefined, and takes actions according to the following cases.*

(1) *$state(\sigma) = \perp$. If $a^\sigma \leq l^\tau$, pick x^σ fresh, and let $state(\sigma) = w_0$.*
(2) *$state(\sigma) = w_i$ for some $i < \|e\|$.*
    (a) *If $B \restriction \psi(a^\sigma)[s] \neq B \restriction \psi(a^\sigma)[s_0]$, let $state(\sigma) = \perp$ and cancel x^σ.*
    (b) *If $B \restriction \psi(a^\sigma)[s] = B \restriction \psi(a^\sigma)[s_0]$ and there exists $i < \|e\|$ such that $x^\sigma \notin W_{n_i}$, choose i_0 as the least such i and let $state(\sigma) = w_{i_0}$.*
    (c) *Both (a) and (c) fail, let $state(\sigma) = w$ and take the actions in (3) immediately.*
(3) *$state(\sigma) = w$.*
    (a) *If $\psi(a^\sigma) \geq x^\sigma$, cancel x^σ, let $state(\sigma) = \perp$;*
    (b) *If (a) fails and there exist α and k such that α is some $\mathcal{M}$- or $\mathcal{R}$-strategy, $\alpha\hat{}\infty \subseteq \sigma$, $\min\{a^\sigma, x^\sigma\} < \phi^\alpha(d^\alpha(k))$ and $\Theta^\alpha(W^\alpha; k) = 0$, choose $\alpha(\sigma)$ as the longest such α and $k(\sigma)$ as the least such k with respect to $\alpha(\sigma)$, enumerate $d^{\alpha(\sigma)}(k(\sigma))$ in D, setup a link $(\alpha(\sigma), \sigma)$;*
    (c) *If both (a) and (b) fail, enumerate a^σ in A and setup a link (τ, σ) and let $state(\sigma) = c$.*

(4) *state*$(\sigma) = c$. *Let* i_0 *be the least* $i < \|e\|$ *such that* $B_{e_i} \upharpoonright \psi(a^\sigma)[s] = B_{e_i} \upharpoonright \psi(a^\sigma)[s_0]$, *enumerate* x^σ *in* C^τ_{e,i_0} *and let* *state*$(\sigma) = c_{i_0}$.
(5) *state*$(\sigma) = c_i$ *for some* $i < \|e\|$. *Do nothing.*

If *state*$(\sigma) \in \{w, c\}$ or (4) happens, then σ has no outcome; otherwise σ has *state*(σ) as outcome.

2.4. **The tree of strategies.** We may consider $\mathcal{N}^\alpha_k$ as subrequirement $\mathcal{N}_{e,k}$ of $\mathcal{M}_e$ where α is an $\mathcal{M}_e$-strategy, $\mathcal{Q}^\tau_n$ and $\mathcal{R}^\tau_{i,j}$ as subrequirements $\mathcal{Q}_{e,n}$ and $\mathcal{R}_{e,i,j}$ of $\mathcal{P}_e$ where τ is a $\mathcal{P}_e$-strategy, and $\mathcal{S}^\eta_k$ as subrequirement $\mathcal{S}_{e,i,j,k}$ of $\mathcal{R}_{e,i,j}$ where η is an $\mathcal{R}^\tau_{i,j}$ strategy and τ is as above. Hence we may regard Θ^α's, C^τ_i's and Θ^η's as local versions of Θ_e's, $C_{e,i}$'s and $\Theta_{e,i,j}$'s.

Fix a computable bijection f mapping ω onto the collection of all requirements such that

(1) $f^{-1}(\mathcal{M}_e) < f^{-1}(\mathcal{N}_{e,k})$;
(2) $f^{-1}(\mathcal{P}_e) < f^{-1}(\mathcal{Q}_{e,n}), f^{-1}(\mathcal{R}_{e,i,j})$;
(3) $f^{-1}(\mathcal{R}_{e,i,j}) < f^{-1}(\mathcal{S}_{e,i,j,k})$.

Let Λ denote the alphabet

$$\{\infty, 0, \perp, \top\} \cup \{c_i : i \in \omega\} \cup \{w_i : i \in \omega\}$$

with an linear ordering $<_\Lambda$ such that

$$\infty <_\Lambda 0 <_\Lambda c_i <_\Lambda \perp <_\Lambda \top <_\Lambda w_i,\ c_{i+1} <_\Lambda c_i \text{ and } w_{i+1} <_\Lambda w_i.$$

We define the tree of strategies $T \subset \Lambda^{<\omega}$ inductively.

Suppose $\xi \in T$. If ξ is an $\mathcal{N}_{e,k}$- ($\mathcal{Q}_{e,n}$-, $\mathcal{R}_{e,i,j}$- or $\mathcal{S}_{e,i,j,k}$-) strategy, let *top*(ξ) be the longest $\eta \subset \xi$ which is an $\mathcal{M}_e$- ($\mathcal{P}_e$-, $\mathcal{P}_e$-, or $\mathcal{R}_{i,j}$-) strategy. We say η is *injured* at ξ if $\eta \subset \xi$ and either

(1) η is an $\mathcal{M}_e$- or $\mathcal{P}_e$-strategy and there are μ and ν such that $\mu\hat{\ }\infty \subseteq \eta \subset \nu\hat{\ }\perp \subseteq \xi$, ν is some $\mathcal{Q}$-strategy and $\mu = top(\nu)$; or
(2) *top*(η) is defined and injured at ξ.

Suppose $\mathcal{X}$ is a requirement, let $\mathcal{X}(\xi)$ be the longest $\mathcal{X}$-strategy $\zeta \subseteq \xi$ not injured at ξ, or undefined if there is no such strategy. $\mathcal{X}$ is *finished* at ξ if one of the following cases applies

(1) $\mathcal{X}$ is an $\mathcal{M}_e$ or $\mathcal{R}_{e,i,j}$, and either $\alpha = \mathcal{X}(\xi)$ is defined and $\alpha\hat{\ }0 \subseteq \xi$ or there is some $\mathcal{Y} = \mathcal{N}_{e,k}$ or $\mathcal{S}_{e,i,j,k}$ such that $\beta = \mathcal{Y}(\xi)$ is defined and $\beta\hat{\ }\perp \subseteq \xi$;
(2) $\mathcal{X}$ is a $\mathcal{P}_e$, and either $\tau = \mathcal{X}(\xi)$ is defined and $\tau\hat{\ }0 \subseteq \xi$, or there is some $\mathcal{Q}_{e,n}$ such that $\sigma = \mathcal{Q}_{e,n}(\xi)$ is defined and $\sigma\hat{\ }\perp \subseteq \xi$;
(3) $\mathcal{X}$ is an $\mathcal{N}_{e,k}$ or $\mathcal{S}_{e,i,j,k}$ and $\mathcal{M}_e$ or $\mathcal{R}_{e,i,j}$ is finished at ξ;
(4) $\mathcal{X}$ is a $\mathcal{Q}_{e,n}$, and either $\mathcal{P}_e$ is finished at ξ, or there is some $\mathcal{Q}_{e,n'}$ and i such that $\sigma = \mathcal{Q}_{e,n'}(\xi)$ is defined, $n_i = n'_i$ and $\sigma\hat{\ }o_i \subseteq \xi$ $(o = w, c)$.

Otherwise $\mathcal{X}$ is *unfinished* at ξ. Furthermore, $\mathcal{X}$ is *satisfied* at ξ if either $\mathcal{X}(\xi)$ is defined or $\mathcal{X}$ is finished at ξ. Otherwise $\mathcal{X}$ is *unsatisfied* at ξ.

Label ξ with the $\mathcal{X}$ such that $f^{-1}(\mathcal{X})$ is the least among the unsatisfied ones, and

(1) If $\mathcal{X}$ is some $\mathcal{M}$, $\mathcal{P}$ or $\mathcal{R}$, let $\xi\hat{\ }\infty, \xi\hat{\ }0 \in T$;
(2) If $\mathcal{X}$ is some $\mathcal{N}$ or $\mathcal{S}$, let $\xi\hat{\ }\perp$ and $\xi\hat{\ }\top \in T$;
(3) If $\mathcal{X}$ is some $\mathcal{Q}_{e,n}$, let $\xi\hat{\ }\perp$, $\xi\hat{\ }w$, $\xi\hat{\ }c$, $\xi\hat{\ }w_i$ and $\xi\hat{\ }c_i \in T$ for $i < \|e\|$.

The following properties of T follow immediately from above.

Lemma 2.4. *Suppose P is an infinite path of T, $\mathcal{X}$ an requirement. Then there is a finite $\xi \subset P$ such that $\mathcal{X}$ is satisfied at η for any finite η such that $\xi \subseteq \eta \subset P$. Hence $\mathcal{X}(P) = \mathcal{X}(\xi)$ is well-defined.*

Lemma 2.5. *Suppose P is an infinite path of T. If $\mathcal{Q}_{e,n}P$ is defined for some e and infinitely many n, then $\mathcal{Q}_{e,n}P\hat{}\perp \not\subset P$ for any such n. Moreover, there is an $i < \|e\|$ such that*

$$(\forall j)(\exists n)(\exists o \in \{w, c\})(n_i = j \wedge \mathcal{Q}_{e,n}(P) \text{ is defined } \wedge \mathcal{Q}_{e,n}(P)\hat{}o_i \subset P).$$

For strategies $\zeta, \eta \in T$, we say that ζ *dominates* η or η *subjects to* ζ iff

(1) $\zeta \subset \eta$, or
(2) there are a common initial segment ξ and letters $o_1 <_\Lambda o_2$ with $\xi\hat{}o_1 \subseteq \zeta$ and $\xi\hat{}o_2 \subseteq \eta$.

At every stage s in the construction, we define an finite approximation TP_s of the *true path* $TP = \liminf_s TP_s$. TP_s is the union of strategies on the tree which are *accessible*, i.e. act, at stage s.

We say that a parameter p (or $p[s']$) *becomes defined at stage* s if p is undefined at stage $s-1$ and is defined at stage s (and is never cancelled between s and $s' \geq s$), or *becomes undefined* if the reverse happens. And we say that $p[s']$ *becomes defined by* ξ *at stage* s, if $s < s'$, ξ is accessible at stage s, p becomes defined at the moment that ξ acts and does not become undefined between s and s'. Or we say that $p[s']$ *becomes undefined by* ξ *at stage* s if the reverse happens.

2.5. **Parameters and Conventions.** We sum up parameters assigned to strategies.

For α an $\mathcal{M}$- or $\mathcal{R}$-strategy, there are

(1) The length of agreement l^α;
(2) A c.e. functional Θ^α to be built;
(3) An official flip point $d^\alpha(k)$ for each k.

For β an $\mathcal{N}_k^\alpha$- or $\mathcal{S}_k^\alpha$-strategy, there is a personal flip point $d^\beta(k)$.

For τ a $\mathcal{P}_e$-strategy, there are

(1) The length of agreement l^τ;
(2) $\|e\|$ many c.e. sets to be built, namely $C_0^\tau, C_1^\tau, \ldots, C_{\|e\|-1}^\tau$.

For σ a $\mathcal{Q}^\tau$-strategy, there are an agitator a^σ, a witness x^σ and $state(\sigma)$.

Given an arbitrary strategy ξ, if it is initialized then all of its parameters and links with one end being ξ are cancelled, i.e. become undefined. But there is an exception, that if ξ is a $\mathcal{Q}$-strategy then $state(\xi)$ is set to be $\perp$.

2.6. **Construction.** Stage 0. Let all c.e. sets and functionals to be constructed be empty, all parameters be undefined and initial states of all $\mathcal{Q}$-strategies are $\perp$.

Stage $s > 0$. Let $\emptyset$ be accessible. Suppose ξ is accessible let $s_0 < s$ be the latest stage such that ξ is accessible at s_0 and never initialized between s_0 and s. We take actions according to the following cases.

Case 1, ξ is an $\mathcal{M}$- or $\mathcal{R}$-strategy.

Subcase 1.1, s is ξ-expansionary.

For each k such that $W \upharpoonright \phi(d^\xi(k))[s] \neq W \upharpoonright \phi(d^\xi(k))[s_0]$, cancel $d^\xi(k)$.

If there is a link (ξ, σ), let σ be accessible and cancel the link. Otherwise let $\xi\hat{}\infty$ be accessible.

Subcase 1.2, s is not ξ-expansionary. Let $\xi\hat{}0$ be accessible.

Case 2, ξ is a $\mathcal{P}_e$-strategy.

Subcase 2.1, s is ξ-expansionary.

For any $\mathcal{Q}$-strategy σ such that $top(\sigma) = \xi$, $state(\sigma) = w$ and $x^\sigma \leq \psi_e(a^\sigma)$, cancel any link with one end being σ.

If there is a link (ξ, σ), let σ be accessible and cancel the link. Otherwise let $\xi\hat{}\infty$ be accessible.

Subcase 2.1, s is not ξ-expansionary. Let $\xi\hat{}0$ be accessible.

Case 3, ξ is a $\mathcal{Q}_{e,n}$-strategy. Let $\tau = top(\xi)$.

If there exists $i < \|e\|$ such that $C_i^\tau \cap W_{n_i} \neq \emptyset$, let i_0 be the greatest such i, let $sate(\xi) = c_{i_0}$ and $\xi\hat{}c_{i_0}$ be accessible.

Otherwise run Procedure 2.3. If ξ has no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

Case 4, ξ is an $\mathcal{N}$- or $\mathcal{S}$-strategy. Run Procedure 2.2. If (2)(a), (2)(d) or (3) of Procedure 2.2 happens, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

If an outcome o is determined and $\xi\hat{}o = s$, let $TP_s = \xi$. If TP_s is defined, we end stage s immediately by taking the following actions.

(I) If TP_s is some $\mathcal{Q}$-strategy and $state(TP_s) = w$, then initialize all strategies subjecting to but not extending TP_s.

(II) Otherwise initialize all strategies subjecting to TP_s.

2.7. **Verifications.** First of all, we study behaviors of flip points.

Lemma 2.6. *α is an $\mathcal{M}_e$–strategy, β is an $\mathcal{N}_k^\alpha$–strategy extending $\alpha\hat{}\infty$.*
(i) If $\Theta^\alpha(A, W_e; k)[s]$ is defined then $\phi_e(d^\alpha(k))[s] < \theta^\alpha(k)[s]$.
(ii) If σ is some $\mathcal{Q}$-strategy extending $\beta\hat{}\top$, $\beta\hat{}\top$ is accessible at s and $a^\sigma[s]$ (or $x^\sigma[s]$) is defined, then $a^\sigma[s]$ (or $x^\sigma[s]$) $> \phi_e(d^\alpha(k))[s]$.
(iii) If σ is some $\mathcal{Q}$-strategy extending $\beta\hat{}\bot$, $\beta\hat{}\bot$ is accessible at s, $\Theta^\alpha(W_e; k)[s]$ converges and $a^\sigma[s]$ (or $x^\sigma[s]$) is defined, then either $a^\sigma[s]$ (or $x^\sigma[s]$) $> \phi_e(d^\alpha(k))[s]$ or $d^\alpha(k)[s] > d^\beta(k)[s]$.
(iv) Suppose σ is some $\mathcal{Q}$-strategy subjecting to β. If σ enumerates some d in D at s and $d^\beta(k)[s]$ is defined, then $d > d^\beta[s]$.

Proof. During the proof, we occasionally omit α and β from the superscripts

(i) Let $s_0 \leq s$ be the earliest stage such that $d^\alpha(k)[s_0]$ is defined and never cancelled between s_0 and s. Then (i) holds at s_0 by (3)(b) of Procedure 2.2.

Let $s_0 < s_1 < \ldots < s_n (\leq s)$ be all α-expansionary stages. Assume (i) holds at s_i and let $u_i = \phi_e(d^\alpha(k))[s_i]$.

If $s_i + m < s_{i+1}$ or s and $\Theta(W_e; k)[s_i + m]$ converges then

$$(W_e[s_i + m] - W_e[s_i]) \upharpoonright u_i \subseteq (W_e[s_i + m] - W_e[s_i]) \upharpoonright \theta(k)[s_i] = \emptyset.$$

Moreover $(A[s_i + m] - A[s_i]) \upharpoonright u_i = \emptyset$ because elements in $A[s_i + m] - A[s_i]$ are contributed by strategies subjecting to $\alpha\hat{}\infty$. Hence $\phi_e(d^\alpha(k))[s_i + m] = u_i < \theta(k)[s_i] = \theta(k)[s_i + m]$.

Since $d^\alpha(k)$ is not cancelled at s_{i+1}, $(W_e[s_i + m] - W_e[s_i]) \upharpoonright u_i = \emptyset$. Moreover, nothing $\leq u_i$ could be enumerated in A at s_{i+1} and $d^\alpha(k)[s_{i+1}] \leq d^\alpha(k)[s_i]$. Hence (i) holds.

(ii) Let $s_0 \leq s$ be the earliest stage such that $\beta\hat{}\top$ is accessible at s_0 and never initialized between s_0 and s. Then $d^\beta(k)[s] = d^\beta(k)[s_0]$ and $(W_e[s] - W_e[s_0]) \upharpoonright \phi_e(d^\beta(k))[s_0] = \emptyset$. Let $d_0 = d^\beta(k)[s_0]$.

All elements of $A[s] - A[s_0]$ are chosen as agitators of $\mathcal{Q}$-strategies at stages not earlier than s_0 and thus greater than $\phi_e(d_0)[s_0]$. Hence $\phi_e(d_0)[s] = \phi_e(d_0)[s_0] \leq s_0$. Since $a^\sigma[s]$ (or $x^\sigma[s]$) is also chosen at some stage not earlier than s_0 and $d^\alpha(k)[s] \leq d_0$, $a^\sigma[s]$ (or $x^\sigma[s]$) $> s_0 \geq \phi_e(d^\alpha(k))[s]$.

(iii) Let $d = d^\alpha(k)[s]$, s_0 be the earliest stage such that $d^\alpha(k)[s_0] = d$ and $d^\alpha(k)$ is never cancelled between s_0 and s, β_0 be an $\mathcal{N}_k^\alpha$-strategy such that $d = d^{\beta_0}(k)[s_0]$ and let $u_0 = \phi_e(d)[s_0]$.

By the choice of s_0 and an argument similar to that in the proof of (i), $(A, W_e)[s] \upharpoonright u_0 = (A, W_e)[s_0] \upharpoonright u_0$ and $\phi_e(d)[s] = u_0$.

If $d^\beta(k)[s] = d$ then $\beta = \beta_0$ and $\beta\hat{}\top$ is accessible at s. This contradicts the assumption of (iii).

If $d^\beta(k)[s] > d$ then $d^\beta[s]$ becomes defined after s_0, and so do $a^\sigma[s]$ (or $x^\sigma[s]$). Hence $a^\sigma[s]$ (or $x^\sigma[s]$) $> u_0 = \phi_e(d)[s]$.

(iv) Let $s_0 \leq s$ be the latest stage at which β is accessible, then $d^\beta(k)[s] = d^\beta(k)[s_0]$. Let α' be some $\mathcal{M}_{e'}$-strategy and k' be such that $d = d^{\alpha'}(k')[s]$. Assume $d^{\alpha'}(k')[s]$ becomes defined at stage $s_1 \leq s$ by some $\mathcal{N}_{k'}^{\alpha'}$-strategy β' and is never canceled between s_1 and s, then $d = d^{\beta'}(k')[s_1]$. By an argument similar to (i), $\phi_{e'}(d)[s] = \phi_{e'}(d)[s_1]$.

Suppose $d = d^{\beta'}(k')[s_1] \leq d^\beta(k)[s] = d^\beta(k)[s_0]$.

If σ dominates β', then so does β. Thus $d^\beta(k)[s_0]$ and $a^\sigma[s]$ become defined after s_1, and $a^\sigma[s] > \phi_{e'}(d)[s_1] = \phi_{e'}(d)[s]$. Hence σ will not enumerate d in D at s, a contradiction.

If σ subjects to β' but does not extend it, then we get a contradiction similar to the previous one.

If $\sigma \supseteq \beta'$, then a contradiction follows from (ii) and (iii). □

By (i) of Lemma 2.6, if $d^\alpha(k)$ is enumerated in D at s then at $s' > s$, the next α-expansionary stage, either $d^\alpha(k)$ is cancelled by α or α is initialized before s'.

Lemma 2.7. *Suppose σ is some $\mathcal{Q}_{e,n}$-strategy accessible at s_0, and $s_1 > s_0$ is the earliest stage at which σ is accessible again. Let $\tau = top(\sigma)$.*
(i) If $state(\sigma)[s_0] = w$ then either σ is initialized between s_0 and s_1, or $state(\sigma)[s_1] = \perp$, or $state(\sigma)[s_1] = w$ and $\alpha(\sigma)[s_1] \subset \alpha(\sigma)[s_0]$, or $state(\sigma)[s_1] = c$.
(ii) If $state(\sigma)[s_0] = c$ then either σ is initialized between s_0 and s, or $state(\sigma)[s_1] = c_i$ for some $i < \|e\|$ and $C^\tau_{e,i} \cap W_{n_i} \neq \emptyset$.

Proof. (i) Suppose σ is not initialized between s_0 and s_1 and $state(\sigma)[s_1] \neq \perp$, then there is a link (α, σ) at stage s_0, s_0 is α-expansionary and $d^\alpha(k(\sigma))[s_0] \in D[s_0] - D[s_0 - 1]$. By the construction, $s_1 > s_0$ is the earliest α-expansionary stage and α is not initialized between s_0 and s_1.

By (i) of Lemma 2.6 and the remark after Lemma 2.6, for each k either $d^\alpha(k)$ is cancelled by α at s or $\phi_e(d^\alpha(k))$ does not increase.

Hence (i) holds by (3) of Procedure 2.3.

(ii) By Procedure 2.3, $l^\tau[s_0] > a^\sigma[s_0]$,

$$\Psi_e(B_e; a^\sigma)[s_0] = 0 \neq 1 = A(a^\sigma)[s_0],$$

and σ setups a link (τ, σ) at stage s_0. By Case 2 of the construction, $s_1 > s_0$ is the earliest τ-expansionary stage and thus

$$\Psi_e(B_e; a^\sigma)[s_1] = 1 \neq 0 = \Psi_e(B_e; a^\sigma)[s_0].$$

Hence for some $i < \|e\|$, $B_{e_i} \upharpoonright \psi_e(a^\sigma)[s_1] \neq B_{e_i} \upharpoonright \psi_e(a^\sigma)[s_0]$ and (ii) holds. □

Let $TP = \liminf_s TP_s$.

Lemma 2.8. *For each* m,
(i) $|TP| \geq m$;
(ii) $TP \upharpoonright m$ *is accessible infinitely often;*
(iii) $TP \upharpoonright m$ *is initialized at most finitely often.*

Proof. We prove (i)(ii) and (iii) simultaneously by induction of m.

For $m = 0$, (i)(ii) and (iii) hold trivially.

Suppose (i)(ii) and (iii) hold for m. Let $\xi = TP \upharpoonright m$ and fix $s_0 > m$ such that ξ is never initialized after stage s_0. We argue by cases.

Case 1, ξ is some $\mathcal{M}_e$- or $\mathcal{R}_{e,i,j}$-strategy.

It suffices to prove that if there are infinitely many ξ-expansionary stages then $\xi\hat{}\infty$ is accessible infinitely often.

Suppose $s_1 > s_0$ is ξ-expansionary and $\xi\hat{}\infty$ is inaccessible at stage s_1. Then there exists a link (ξ, σ). Since ξ will no longer be initialized, σ will not be initialized before next ξ-expansionary stage $s_2 > s_1$.

If the link is canceled before s_2 (because of subcase 2.1 of the construction), then $\xi\hat{}\infty$ is accessible at s_2.

Otherwise, by Lemma 2.7, either $\alpha(\sigma)[s_2] \subset \xi$ or $state(\sigma)[s_2] = c$.

By induction hypothesis and Lemma 2.7, there is $s > s_2$ such that $state(\sigma)[s] = c_i$ for some i. Let s_3 be the least such s, then $TP_{s_3} = \sigma$ and there is no link along TP_{s_3}. Let s_4 be the earliest ξ-expansionary stage after s_3, then $\xi\hat{}\infty$ is accessible.

Case 2, ξ is some $\mathcal{P}_e$-strategy.

It suffices to prove that if there are infinitely many ξ-expansionary stages then $\xi\hat{}\infty$ is accessible infinitely often.

Suppose $s_1 > s_0$ is ξ-expansionary and $\xi\hat{}\infty$ is inaccessible at stage s_1. Then there exists a link (ξ, σ) and $state(\sigma) = c$. By Lemma 2.7, the link is canceled at s_1 and $\xi\hat{}\infty$ is accessible at next ξ-expansionary stage.

Case 3, ξ is some $\mathcal{Q}_{e,n}$-strategy.

By induction hypothesis, we may assume that $a^\xi[s] = a^\xi[s_0]$ for $s > s_0$.

If ξ has $\perp$ as outcome for infinitely often, then by Case 3 of the construction, Procedure 2.3 and Lemma 2.7, $state(\xi)[s] \notin \{c_{\|n\|}, \ldots, c_1, c_0, c\}$ for $s > s_0$. The lemma holds because by (I) of the construction, $\xi\hat{}\perp$ will not be initialized when $TP_s = \xi$ and $state(\xi)[s] = w$.

If ξ has some c_i as outcome at some stage $s > s_0$, then by Case 3 of the construction and (5) of Procedure 2.3, ξ eventually has some c_{i_0} as outcome where $i_0 \geq i$. Otherwise, ξ eventually has some w_i as outcome. In either case the lemma holds obviously.

Case 4, ξ is some $\mathcal{N}_{e,k}$- or $\mathcal{S}_{e,i,j,k}$-strategy. Let $\alpha = top(\xi)$. We only prove the case for $\mathcal{N}_{e,k}$ since the other case is similar.

If $TP_{s_1} = \xi$ at $s_1 > s_0$, then the first clause of (3)(d) of Procedure 2.2 happens at s_1. Let $s_2 > s_1$ be the next α-expansionary stage, by the remark after Lemma 2.6 $d^\alpha(k)$ is cancelled by α at this stage. Let $s_3 \geq s_2$ be the earliest stage at which ξ is accessible again, then either $\xi\hat{}\perp$ is accessible or $\Theta^\alpha(W_e; k)[s_3] = 1$ and $\xi\hat{}\top$ is accessible. □

Lemma 2.9. *If* β *is an* $\mathcal{N}_{e,k}$*- or* $\mathcal{S}_{e,i,j,k}$*-strategy on* TP*, then* d^β *is fixed eventually.*

Proof. Let $\alpha = top(\beta)$ and s_0 be the stage such that β is never initialized after s_0. We will only prove the case that β is $\mathcal{N}_{e,k}$-strategy since the other is similar and easier.

By the construction, d^β could be canceled only if it were enumerated in D previously. Moreover, d^β could be enumerated in D after s_0 only if $K(k) = 1 \neq 0 = \Theta^\alpha(A, W_e; k)$ or by some σ such that $\alpha\hat{}\infty \subseteq \sigma \subset \beta$.

Note that the former situation could happen at most once. For the latter, if $\sigma\hat{}\bot$ is not on TP then σ could enumerate d^β in D at most finitely often.

Assume $\sigma\hat{}\bot \subset TP$. If $\tau \subset \alpha\hat{}\infty \subseteq \sigma$, then by the definition of T, $\beta \subset \sigma$. By Lemma 2.6 (iv), σ will never enumerate d^β in D.

If $\alpha\hat{}\infty \subseteq \tau$ and σ enumerates d^β in D at $s_1 > s_0$, then σ setups a link (α, σ) at s_1. From then on τ is skipped and the enumeration B^τ will never change until later $state(\sigma) = c$ and a link (τ, σ) is setup. By Lemma 2.7 and the choice of s_0, $\sigma\hat{}c_i \subset TP$ for some i. This contradicts the assumption that $\sigma\hat{}\bot \subset TP$. □

If α is an $\mathcal{M}_e$-strategy on TP and α is never initialized after s_0, then $\Theta^\alpha = \bigcup_{s>s_0} \Theta^\alpha[s]$ is consistent.

If β is an $\mathcal{N}_k^\alpha$-strategy on TP, then by the lemma above, $d^\beta(k)$ is fixed eventually. If in addition $\beta\hat{}\top \subset TP$, then $d^\alpha(k)$ is eventually fixed too by (ii) of Lemma 2.6, and $\Theta^\alpha(W_e; k) = K(k)$ by Case 4 in the proof of Lemma 2.8 and (3)(d) of Procedure 2.2.

Thus we get the following.

Lemma 2.10. *$\mathcal{M}_e$ is satisfied for every e.*

Now we turn to $\mathcal{P}_e$.

If τ is a $\mathcal{P}_e$-strategy on TP and never initialized after s_0, then $C^\tau_{e,i} = \bigcup_{s>s_0} C^\tau_{e,i}[s]$ is c.e. for $i < \|e\|$.

If $\tau\hat{}0 \subset TP$ then $C^\tau_{e,i}$ is finite for $i < \|e\|$. Otherwise, to determine whether $x \in C^\tau_{e,i}$ for an arbitrary x and $i < \|e\|$, let $s > s_0$ be the earliest τ-expansionary stage such that $B_{e_i} \upharpoonright x = B^\tau_{e_i}[s] \upharpoonright x$, then $x \in C^\tau_{e,i}$ iff $x \in C^\tau_{e,i}[s]$. Hence we establish $C^\tau_{e,i} \leq_{\mathrm{T}} B_{e_i}$ for $i < \|e\|$.

Suppose $A = \Psi_e(B_e)$, and let σ be a $\mathcal{Q}^\tau_{e,n}$-strategy on TP. Then the satisfaction of $\mathcal{Q}^\tau_{e,n}$ follows from Lemma 2.7. The argument for $\mathcal{R}_{e,i,j}$'s is similar to that for Lemma 2.10. Hence we get the next lemma and finish the proof of Theorem 2.1.

Lemma 2.11. *$\mathcal{P}_e$ is satisfied for every e.*

3. $[\mathbf{PC}] \not\subseteq [\mathbf{NB} \cup \mathbf{NCup}]$

Yu and Yang showed that $I = [\mathbf{NB} \cup \mathbf{NCup}] \subset \mathbf{M}$ in [11]. In this section, we will prove the following.

Theorem 3.1. *There is a plus cupping degree $\mathbf{a} \notin I$.*

We construct a c.e. set A satisfying the plus cupping requirements

$$\mathcal{M}_e : W_e = \Phi_e(A) \Rightarrow W_e \leq_{\mathrm{T}} \emptyset \text{ or } W_e \text{ is cupping},$$

and the requirements guaranteeing $\mathbf{deg}(A) \notin [\mathbf{NB} \cup \mathbf{NCup}]$

$$\mathcal{P}_e : A = \Psi_{e_c}(X_e, Y_{e_{c-1}}) \Rightarrow (\exists i < c-1)(X_{e_i} \text{ is bounding}) \text{ or } Y_{e_{c-1}} \text{ is cupping}$$

where X_e is the abbreviation of the tuple $(X_{e_0}, \ldots, X_{e_{c-2}})$ and $c = \|e\|$.

We will arrange the construction on a tree of strategies. Every finite path of the tree is a strategy serving $\mathcal{M}_e, \mathcal{P}_e$ or their subrequirements introduced later. At every stage s we will define an ascending finite sequence of strategies, called *accessible* strategies, and the union of this sequence, TP_s. We will guarantee that there is an infinite leftmost path $TP = \liminf_s TP_s$ and every strategies on this path is eligible to win.

During the construction, we will in addition build a c.e. set D for some diagonalization purposes which will be clear.

3.1. **$\mathcal{M}$-strategies.** We follow the technique originally developed by Harrington [3] and refined by Fejer and Soare [2].

Suppose α is an $\mathcal{M}_e$-strategy, let l^α the length of agreement between W_e and $\Phi_e(A)$ and α-expansionary stages be defined as usual. If there are at most finitely many α-expansionary stages, α has 0 as outcome; otherwise α has ∞ as outcome.

In the latter case, α will build a c.e. set C^α and a p.r. functional Δ^α such that $K = \Delta^\alpha(W_e, C^\alpha)$, and

$$\mathcal{N}_i^\alpha : D \neq \Gamma_i(C^\alpha) \text{ or } W_e \leq_{\mathrm{T}} \emptyset.$$

From now on we will omit the superscript α in this section.

To define $\Delta(W_e, C; k)$, at the beginning α defines $\Delta(W_e, C; k) = K(k)$ with an arbitrary use. If later k is enumerated in K, α enumerates $\delta(k)$ in C and redefines $\Delta(W_e, C; k) = 1$ with a fresh use.

To make $\mathcal{N}_i^\alpha$, we arrange $\mathcal{N}_i^\alpha$-strategies above $\alpha\hat{}\infty$. If β is an $\mathcal{N}_i^\alpha$-strategy, β picks a fresh *diagonalizer* d^β and a *lifting point* k^β at the beginning and keeps d^β from entering D. From now on we will omit the superscript β in this section.

If $\Gamma_i(C; d) \neq 0$ for ever, then β has outcome 0 and $\mathcal{N}_i^\alpha$ is satisfied since $d \notin D$. Otherwise at some α-expansionary stage s_0 $\Gamma_i(C; d) = 0$, β will try to clear $\delta(k')$ for $k' \geq k$ for preserving the computation $\Gamma_i(C; d) = 0$. If this is achieved, β will enumerate d in D and hence win by establishing $D(d) = 1 \neq 0 = \Gamma_i(C; d)$.

To clear δ's, β *opens a gap* by having g as outcome and allowing strategies above $\beta\hat{}g$ to contribute arbitrary numbers in A, and setups a *shortcut* (α, β).

At the next α-expansionary stage $s_1 > s_0$, we will have α *close the gap for β.*

If α finds that $W_e \upharpoonright \delta(k)[s_1] \neq W_e \upharpoonright \delta(k)[s_0]$, then $\Delta(W_e, C; k)$ diverges since no $\mathcal{N}^\alpha$-strategy acts before α does. In this case, α tries to preserve the computation $\Gamma_i(C; d)$ by lifting $\delta(k)$, hence the intention to make $\Delta(W_e, C; k)$ and $K(k)$ agree will not harm the computation $\Gamma_i(C; d)$. Then α *closes the gap successfully* by enumerating d in D. In this case, β will open no more gap.

If α finds that $W_e \upharpoonright \delta(k)[s_1] = W_e \upharpoonright \delta(k)[s_0]$, α tries to preserve the computation $\Phi_e(A; k)[s_1]$ by initializing strategies dominated by but not extending $\beta\hat{}g$. Then α lifts $\delta(k)$ by enumerating $\delta(k)[s_0]$ in C and thus cancelling $\Delta(W_e, C; k')$ for $k' \geq k$. We say that α *closes the gap unsuccessfully*.

In either cases above, α will cancel (α, β). The purpose of using shortcuts is to guarantee validity of the argument below.

If there are infinitely many gaps opened and closed (unsuccessfully), let $(s_m : m \in \omega)$ increasingly enumerate the stages at which β opens a gap. For each m let t_m be the earliest α-expansionary stage after s_m, then the gap opened at s_m is closed by α at t_m. Since $\delta(k)[s_{m+1}] > \delta(k)[s_m]$, $W_e \upharpoonright \delta(k)[s_m]$ is fixed between s_m and t_m while $\Phi_e(A) \upharpoonright \delta(k)[s_m]$ is fixed between t_m and s_{m+1}, W_e is computable if $W_e = \Phi_e(A)$.

Thus we will run no $\mathcal{N}^\alpha$-strategies above $\beta\hat{\ }g$.

However, to guarantee that $\Delta(W_e, C; k)$ converges, we must arrange the distribution of lifting points so that there are at most finitely many $\mathcal{N}^\alpha$-strategies having lifting point less than k' for each k'.

We formally describe the behavior of α at stage s as below. Let

$$s_0 = \max\{s' < s : \alpha \text{ is accessible at } s' \text{ and not initialized between } s' \text{ and } s\}.$$

Procedure 3.2. *There are two cases.*
(i) Case 1, s is not α-expansionary. *Just have 0 as outcome.*
(ii) Case 2, s is α-expansionary. *If in additional there is a shortcut (α, β), then the shortcut is setup at stage s_0, let $k_0 = k^\beta$; otherwise let $k_0 = s$. Let $k_1 = \min\{k < k_0 : \Delta(W_e, C; k) = 0 \neq 1 = K(k)\}$. Whatever α does, let ∞ be the outcome, and if there is a shortcut then it is cancelled immediately after α finishes its jobs at current stage.*

(1) *If k_1 is defined, enumerate $\delta(k_1)$ in C. Redefine $\Delta(W_e, C; k') = K(k')$ for $k' \geq k$ with $\delta(k')$ fresh.*
(2) *From now on assume k_1 is undefined. For $k' < k_0$, if $\Delta(W_e, C; k')$ diverges, define $\Delta(W_e, C; k') = K(k')$ with $\delta(k') = \delta(k')[s_0]$ if s_0 is defined and $\Delta(W_e, C; k')[s_0]$ converges, or with $\delta(k')$ fresh.*
(3) *If $k_0 = k^\beta$ and $W_e[s] \upharpoonright \delta(k)[s_0] \neq (W_e \upharpoonright \delta(k))[s_0]$, then define $\Delta(W_e, C; k') = K(k')$ with $\delta(k')$ fresh for $k' \geq k_0$ and enumerate d^β in D.*
(4) *If $k_0 = k^\beta$ and $W_e[s] \upharpoonright \delta(k)[s_0] = (W_e \upharpoonright \delta(k))[s_0]$, enumerate $\delta(k)[s_0]$ in C if $\delta(k)[s_0]$ is defined, define $\Delta(W_e, C; k') = K(k')$ with $\delta(k') = l^\alpha[s] + k' - k_0$ for $k' \geq k_0$ and initialize $\beta\hat{\ }0$ and strategies subjecting to $\beta\hat{\ }0$.*

We formally describe the behavior of β at stage s as below. Once the outcome is determined, β stops immediately.

Procedure 3.3. *Define k to be fresh if it is undefined.*

(1) *If $\Gamma_i(C; d) = 1 = D(d)$, cancel d.*
(2) *If d is undefined, define it to be fresh.*
(3) *If $\Gamma_i(C; d) \neq 0$, let 0 be the outcome.*
(4) *If $\Gamma_i(C; d) = 0 \neq 1 = D(d)$, let 1 be the outcome.*
(5) *Otherwise $\Gamma_i(C; d) = 0 = D(d)$, setup a shortcut (α, β) and let g be the outcome.*

3.2. **$\mathcal{P}$-strategies.** We follow the proof of Theorem 1.6 in Yu and Yang [11].

Suppose τ is a $\mathcal{P}_e$-strategy, the length of agreement l^τ and the τ-expansionary stages are defined as usual. If there are at most finitely many τ-expansionary stages, τ has 0 as outcome; otherwise τ has ∞ as outcome.

In the latter case, τ will construct $2c - 1$ ($c = \|e\|$) c.e. sets

$$M^\tau_{0,0}, M^\tau_{0,1}, \ldots, M^\tau_{c-2,0}, M^\tau_{c-2,1}, Z^\tau$$

and one p.r. functional Θ^τ so that $M^\tau_{i,0}, M^\tau_{i,1} \leq_{\mathrm{T}} X_{e_i}$ for $i < c - 1$, $K = \Theta^\tau(Y_{c-1}, Z^\tau)$, for $\|n\| = c - 1$ and $(i, j) \in (c - 1) \times 2$

$$\mathcal{Q}^\tau_{n,j} : D \neq \Phi_{n_{c-1}}(Z^\tau) \text{ or } (\exists i < c - 1)(M^\tau_{i,j} \neq \overline{W}_{n_i}), \text{ and}$$

$$\mathcal{R}^\tau_{i,j} : \Phi_j(M^\tau_{i,0}) = \Phi_j(M^\tau_{i,1}) \text{ is total} \Rightarrow \Phi_j(M^\tau_{i,0}) \leq_{\mathrm{T}} \emptyset, \text{ for } (i, j) \in \omega^2.$$

From now on in this subsection, we will drop the superscript τ and occasionally also drop the subscripts such as e and e_i.

To define $\Theta(Y,Z;k)$, at the beginning τ defines $\Theta(Y,Z;k) = K(k)$ with an arbitrary use. If k is enumerated in K later, τ enumerates $\theta(k)$ in Z and redefines $\Theta(Y,Z;k) = 1$ with a fresh use.

To satisfy $\mathcal{Q}^\tau$'s and $\mathcal{R}^\tau$'s, we arrange ζ's for $\mathcal{Q}^\tau$'s and η's for $\mathcal{R}^\tau$s above $\tau\hat{}\infty$. As in subsection 2.2, we will arrange $\mathcal{Q}^\tau$ so that on every infinite path extending $\tau\hat{}\infty$ we could make either $D \neq \Phi_j(Z^\tau)$ for all j or $M^\tau_{i,0}, M^\tau_{i,1} \neq \overline{W}_j$ for some i and all j.

Suppose ζ is a $\mathcal{Q}^\tau_{n,0}$-strategy. At the beginning ζ picks a fresh *lifting point* k, a fresh *diagonalizer* d and a fresh *agitator* a, and keeps d and a from entering D or A respectively. ζ makes $\theta(k) > \psi(a)$ by lifting $\theta(k)$ whenever $\psi(a)$ grows.

If ζ finds that the computation $\Psi(X,Y;a)$ changes infinitely often, then it has $\perp$ as outcome indicating that $\Psi(X,Y;a)$ diverges. We will have neither $\mathcal{Q}$- nor $\mathcal{R}$-strategies above $\zeta\hat{}\perp$.

If $\Psi(X,Y;a)$ is eventually fixed, ζ defines a *witness* $x > \psi(a)$ and waits for $\Phi(Z;d) = 0$ and $x \in \bigcup_{i<c-1} W_{n_i}$. If the former does not happen, ζ has 0 as outcome indicating $D(d) = 0 \neq \Phi(Z;d)$. If the latter does not happen, ζ has outcome $w_{i,0}$ (suppose $x \notin W_{n_i}$) indicating $M_{i,0}(x) = 0 \neq 1 = \overline{W}_{n_i}(x)$.

If at some stage s_0, $\Phi(Z;d) = 0$ and $x \in W_{n_i}$ for all $i < c-1$, ζ will try to clear $\theta(k)$ for preserving the computation $\Phi(Z;d) = 0$. If this is achieved, ζ will enumerate d in D and establish $D(d) = 1 \neq \Phi(Z;d)$.

To this end, ζ will enumerate a in A and setup a link (τ,ζ). At next τ-expansionary stage $s_1 > s_0$, one of $X_{e_0},\dots,X_{e_{c-2}}$ and Y must have been changed below $\psi(a)[s_0]$. The control will be passed immediately from τ to ζ and the link will be cancelled, i.e., the link will be *travelled*.

If Y does, then $\Theta(Y,Z;k)$ diverges. In this case ζ will success in clearing $\theta(k)$ from $\phi(d)$. ζ clears $\theta(k)$ by defining $\Theta(Y,Z;k)$ with $\theta(k)$ fresh, then enumerates d in D and have 1 as outcome.

If some X_{e_i} does, ζ will enumerate x in $M_{i,0}$ and have $m_{i,0}$ as outcome. In this case ζ will win by establishing $M_{i,0}(x) = 1 \neq 0 = \overline{W}_{n_i}(x)$.

The purpose of using links is to guarantee $M_{i,0} \leq_{\mathrm{T}} X_{e_i}$ by permitting.

We formally describe the actions of τ at stage s as below. Let s_0 be defined as before Procedure 3.2 (with τ in place of α).

Procedure 3.4. *There are two cases.*
(i) Caes 1, s is not τ-expansionary. *Just let 0 be the outcome.*
(ii) Caes 2, s is τ-expansionary. *If there is a link (τ,ζ), then it is setup by ζ at stage s_0, let $k_0 = k^\zeta$; otherwise let $k_0 = s$. Let $k_1 = \min\{k < k_0 : \Theta(Y,Z;k) = 0 \neq 1 = K(k)\}$.*

(1) *If k_1 is defined, enumerate $\theta(k_1)$ in Z and redefine $\Theta(Y,Z;k') = K(k')$ with $\theta(k')$ fresh for $k' \geq k_1$; if there is a link (τ,ζ), travel and cancel it.*
(2) *From now on, assume k_1 is undefined. For $k' < k_0$, if $\Theta(Y,Z;k')$ diverges define $\Theta(Y,Z;k') = K(k')$ with $\theta(k') = \theta(k')[s_0]$ if s_0 is defined and $\Theta(Y,Z;k')[s_0]$ converges, or with $\theta(k')$ fresh.*
(3) *If there is no link, let ∞ be the outcome and stop. Otherwise assume that there is a link (τ,ζ), travel and cancel the link.*

We formally describe the actions of ζ at stage s as below.

Procedure 3.5. *There are two cases.*
(i) Case 1, the link (τ, ζ) is travelled. *Suppose the link is setup at stage $s_0 < s$. Take actions according to the following subcases.*

(1) *If $K[s] \upharpoonright k \neq K[s_0] \upharpoonright k$ then cancel a, d and x.*
(2) *If $Y[s] \upharpoonright \psi(a)[s_0] \neq (Y \upharpoonright \psi(a))[s_0]$, then $\Theta(Y, Z; k)[s-1]$ diverges, define $\Theta(Y, Z; k') = K(k')$ with $\theta(k')$ fresh for $k' \geq k$ and enumerate d in D.*
(3) *Otherwise, there is some $i < c-1$ such that $X_{e_i}[s] \upharpoonright \psi(a)[s_0] \neq (X_{e_i} \upharpoonright \psi(a))[s_0]$. Let i_0 be the greatest such i, enumerate x in $M_{i_0,j}$.*

(ii) Case 2, otherwise. *Check the followings one by one. Once an outcome is determined, ζ stops immediately.*

(1) *If k is undefined, define it to be fresh.*
(2) *If $D(d) = 1 = \Phi_{n_{c-1}}(Z; d)$, cancel a, d and x.*
(3) *If $D(d) = 1 \neq \Phi_{n_{c-1}}(Z; d)$, let 1 be the outcome. If $M_{i,j} \cap W_{n_i}$ is not empty for some $i < c-1$, let i_0 be the greatest i and let $m_{i_0,j}$ be the outcome.*
(4) *If a is undefined, define it to be fresh. If $l^\tau < a$, stop.*
(5) *Otherwise if $\psi(a) \geq \theta(k)$, enumerate $\theta(k)$ in Z and redefine $\Theta(Y, Z; k')$ with $\theta(k')$ fresh for $k' \geq k$ (if $\Theta(Y, Z; k)[s-1]$ is defined), cancel d and x, let $\perp$ be the outcome.*
(6) *If d and x are undefined, define them to be fresh. If $D(d) = 0 \neq \Phi_{n_{c-1}}(Z; d)$, let 0 be the outcome. If there is some $i < c-1$ such that $x \notin W_{n_i}$, let i_0 be the least such i and let $w_{i_0,j}$ be the outcome.*
(7) *Otherwise enumerate a in A and setup a link (τ, ζ).*

The $\mathcal{R}$-strategies η's act in the same way as typical minimal pair constructions. We define l^η the length of agreement between $\Phi_j(M_{i,0})$ and $\Phi_j(M_{i,1})$ and η-expansionary stages as usual. Each η has two outcomes, namely ∞ indicating there are infinitely many η-expansionary stages, and 0 indicating there are at most finitely many such stages. We refer the readers to XIV.3.2 in Soare [10] for details.

3.3. **Conflicts.** Different $\mathcal{M}$-strategies do not injure each other, because they never intend to change A and they build local Δ's and C's. Neither do different $\mathcal{N}^\alpha$-strategies above a certain $\mathcal{M}_e$-strategy α injure each other, because none of them intend to change C^α.

If β is some $\mathcal{N}_i^\alpha$-strategy, then the intention of β to preserve $C^\alpha \upharpoonright \gamma_i(d^\beta)$ may be injured by the intention of α to define $\Delta^\alpha(W_e, C^\alpha; k) = K(k)$ for $k < k^\beta$, and the intention of β to lift $\delta^\alpha(k^\beta)$ may injure the intention of α to make $\Delta^\alpha(W_e, C^\alpha; k)$ converge. The first conflict is solved by guaranteeing that k^β is eventually fixed, hence it could happen at most finitely often (this is also the solution of similar conflicts between $\mathcal{P}$-strategies and $\mathcal{Q}$-strategies). To solve the second conflict, note that β intends to lift $\delta^\alpha(k^\beta)$ infinitely often only if it opens infinitely many gaps. In this case we will make $W_e \leq_{\mathrm{T}} \emptyset$ hence will not worry about the definition of Δ^α. Otherwise we arrange the distribution of lifting points so that each k is used as a lifting point by at most one $\mathcal{N}^\alpha$-strategy. This is achieved by the first sentence of Procedure 3.3. Hence $\delta^\alpha(k)$ will not be lifted for ever if every $\mathcal{N}^\alpha$-strategy lifts its lifting point at most finitely often.

Now the intention of α to preserve $\Phi_e(A; k^\beta)$ when unsuccessfully closing a gap opened by β could be injured by some $\mathcal{Q}^\tau$-strategy ζ where τ is some $\mathcal{P}$-strategy since ζ may enumerate its agitator in A. The solution is to initialize ζ if it subjects to $\beta\hat{\ }0$ or it is $\beta\hat{\ }0$. Hence α will succeeded in preserving $\Phi_e(A; k^\beta)$ if $\mathcal{Q}$-strategies dominating $\beta\hat{\ }g$ are never accessible later, since A can be freely changed above $\beta\hat{\ }g$. This is already incorporated by (ii)(4) of Procedure 3.2.

The last kind of conflicts is between $\mathcal{R}^\tau_{i,j}$-strategies η's and $\mathcal{Q}^\tau_{n,j'}$-strategies ζ's. The solution is to allow at most one side of $\Phi_j(M^\tau_{i,0}) = \Phi_j(M^\tau_{i,1})$ be destroyed between η-expansionary stages. To this end we will run no more strategies at a stage once (i)(3) of Procedure 3.5 happens.

3.4. **Parameters.** We sum up parameters associated with strategies.

For α an $\mathcal{M}$-strategy, there are the length of agreement l^α, a c.e. set C^α to be built and a p.r. functional Δ^α.

For β an $\mathcal{N}^\alpha$-strategy, there are a diagonalizer d^β and a lifting point k^β.

For τ a $\mathcal{P}_e$-strategy, there are

1. The length of agreement l^τ;
2. $2\|e\|-1$ many c.e. sets $M^\tau_{0,0}, M^\tau_{0,1}, \dots, M^\tau_{\|e\|-2,0}, M^\tau_{\|e\|-2,1}$ and Z^τ;
3. A p.r. functional Θ^τ.

For ζ a $\mathcal{Q}^\tau$-strategy, there are a lifting point k^ζ, a diagonalizer d^ζ, an agitator a^ζ and a witness x^ζ.

For η an $\mathcal{R}^\tau$-strategy, there is the length of agreement l^η.

Assume ξ is an arbitrary strategy. If it is initialized then all of its parameters, and shortcuts or links with one end being ξ are cancelled.

3.5. **The tree of strategies.** We may consider $\mathcal{N}^\alpha_i$ as a subrequirement $\mathcal{N}_{e,i}$ of $\mathcal{M}_e$ where α is a $\mathcal{M}_e$-strategy, and $\mathcal{Q}^\tau_{n,j}$ and $\mathcal{R}^\tau_{i,j}$ as subrequirements $\mathcal{Q}_{e,n,j}$ and $\mathcal{R}_{e,i,j}$ of $\mathcal{P}_e$ where τ is a $\mathcal{P}_e$-strategy. Hence $C^\alpha, \Delta^\alpha, M^\tau_{i,j}$ and Z^τ, Θ^τ may be taken as local versions of $C_e, \Delta_e, M_{e,i,j}$ and Z_e, Θ_e respectively.

Let Λ be the set of outcomes

$$\{\infty, 1, g, \perp, 0\} \cup \{m_{i,j} : (i,j) \in \omega \times 2\} \cup \{w_{i,j} : (i,j) \in \omega \times 2\}$$

with a computable linear ordering $<_\Lambda$ such that

$$\infty <_\Lambda 1 <_\Lambda g <_\Lambda m_{i,j} <_\Lambda \perp <_\Lambda w_{i,j} <_\Lambda 0,$$

$$m_{i+1,1} <_\Lambda m_{i+1,1} <_\Lambda m_{i,0} <_\Lambda m_{i,1} \text{ and } w_{i+1,1} <_\Lambda w_{i+1,1} <_\Lambda w_{i,0} <_\Lambda w_{i,1}.$$

Fix a computable bijection f mapping ω onto the collection of all requirements and subrequirements such that $f^{-1}(M_e) < f^{-1}(N_{e,k})$, and $f^{-1}(P_e) < f^{-1}(Q_{e,n,j}), f^{-1}(R_{e,i,j})$. We inductively define T the tree of strategies as a computable subset of $\Lambda^{<\omega}$.

Let $\emptyset \in T$. If $\xi \in T$, we say that a requirement $\mathcal{O}$ is *finished at* ξ if and only if one of the followings applies

1. $\mathcal{O}$ is $\mathcal{M}_e$ and either there is an $\mathcal{M}_e$-strategy $\alpha \subset \alpha\hat{\ }0 \subseteq \xi$ or there is an $\mathcal{N}_{e,i}$-strategy $\beta \subset \beta\hat{\ }g \subseteq \xi$.
2. $\mathcal{O}$ is $\mathcal{P}_e$ and either there is a $\mathcal{P}_e$-strategy $\tau \subset \tau\hat{\ }0 \subseteq \xi$ or there is a $\mathcal{Q}_{e,n,j}$-strategy $\zeta \subset \zeta\hat{\ }\perp \subseteq \xi$.
3. $\mathcal{O}$ is $\mathcal{N}_{e,i}$ ($\mathcal{Q}_{e,n,j}$ or $\mathcal{R}_{e,i,j}$) and $\mathcal{M}_e$ ($\mathcal{P}_e$) is finished at ξ.
4. $\mathcal{O}$ is $\mathcal{Q}_{e,n,j}$ and there is a $\mathcal{Q}_{e,n',j'}$-strategy $\zeta \subset \xi$ such that either $n_{\|n\|} = n'_{\|n'\|}$ and $\zeta\hat{\ }o \subseteq \xi$ for $o \in \{0,1\}$, or $j = j'$, $n_i = n'_i$ and $\zeta\hat{\ }o_{i,j} \subseteq \xi$ for some $i \leq \|n\|$ and $o \in \{w, m\}$.

We say that $\mathcal{O}$ is *satisfied at* ξ if either $\mathcal{O}$ is finished at ξ or there is an $\mathcal{O}$-strategy $\xi' \subset \xi$; otherwise we say that $\mathcal{O}$ is *unsatisfied at* ξ.

We assign the unique $\mathcal{O}$ to ξ such that $f^{-1}(\mathcal{O})$ is the least among the requirements unsatisfied at ξ.

If ξ is some $\mathcal{M}$-, $\mathcal{P}$- or $\mathcal{R}$-strategy, let $\xi\hat{}\infty$ and $\xi\hat{}0 \in T$; if ξ is an $\mathcal{N}$-strategy, let $\xi\hat{}1, \xi\hat{}g$ and $\xi\hat{}0 \in T$; if ξ is a $\mathcal{Q}_{e,n,j}$-strategy, let $\xi\hat{}1, \xi\hat{}\bot, \xi\hat{}0, \xi\hat{}m_{i,j}$ and $\xi\hat{}w_{i,j} \in T$ where $i < \|n\|$.

Furthermore, if ξ is an $\mathcal{N}_{e,i}$-strategy, let $top(\xi)$ be the unique $\mathcal{M}_e$-strategy $\alpha \subset \xi$; if ξ is a $\mathcal{Q}_{e,n,j}$- or $\mathcal{R}_{e,i,j}$-strategy, let $top(\xi)$ be the unique $\mathcal{P}_e$-strategy $\tau \subset \xi$.

We will use some terminologies defined in subsection 2.4.

3.6. **Construction.** Stage 0. Let all parameters associated with all strategies be undefined, and all c.e. sets and p.r. functionals to be built be empty.

Stage $s > 0$. Let $\emptyset$ be accessible. If ξ is accessible and $|\xi| = s$, let $TP_s = \xi$. Otherwise we take actions according to the following cases.

Case 1, ξ is an $\mathcal{M}_e$-strategy. Run Procedure 3.2. Let $\xi\hat{}o$ be accessible where o is the outcome.

Case 2, ξ is an $\mathcal{N}_{e,i}$-strategy. Run Procedure 3.3. If there is no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

Case 3, ξ is a $\mathcal{P}_e$-strategy. Run Procedure 3.4. If there is an outcome o let o be accessible; otherwise there is a link (ξ, ζ) at the beginning of s and the last clause of (ii)(3) of Procedure 3.4 happens, let ζ be accessible.

Case 4, ξ is a $\mathcal{Q}_{e,n,j}$-strategy. Let $\tau = top(\xi)$. Run Procedure 3.5. If there is no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

Case 5, ξ is an $\mathcal{R}_{e,i,j}$-strategy. If s is ξ-expansionary, let $\xi\hat{}\infty$ be accessible; otherwise let $\xi\hat{}0$ be accessible.

In addition, once TP_s is defined, we end stage s immediately by initializing all strategies dominated by TP_s.

3.7. **Verification.** First we study an important behavior of $\mathcal{N}$-strategies.

Lemma 3.6. *If α is an $\mathcal{M}_e$-strategy and β is an $\mathcal{N}_i^\alpha$-strategy above $\alpha\hat{}\infty$, then either β is initialized infinitely often or d^β is eventually fixed.*

Proof. During the proof, we occasionally omit α and β from the superscripts.

If β is accessible at most finitely often, then it is trivial that d^β is eventually fixed (including the possibility that it is cancelled at some stage and never becomes defined from then on).

From now on we assume that β is accessible infinitely often and initialized at most finitely often. We may assume in addition that every proper initial segment of β being also some $\mathcal{N}^\alpha$-strategy has its diagonalizer eventually fixed. Let s_0 be such that

(1) β is not initialized after s_0 and $k^\beta = k^\beta[s_0]$;
(2) For all $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and if $k \in K$ then $\Delta(W_e, C; k)[s_0] = 1 = K(k)[s_0]$;
(3) For $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $d^{\beta'} = d^{\beta'}[s_0]$.

If at stage $s > s_0$, d is cancelled by β then $\Gamma_i(C; d) = 1 = D(d)$. Let $s_2 < s$ be the stage at which $d^\beta[s_2 - 1]$ is enumerated in D by α. Then there is a shortcut (α, β) at the beginning of s_2, suppose it is setup by β at stage $s_1 < s_2$. We may assume $s_1 > s_0$ otherwise d could be cancelled at most s_0 many times.

At stage s_1, $\Gamma_i(C;d) = 0 = D(d)$. At the beginning of stage s_2 the computation $\Gamma_i(C;d)$ is same as that at s_1 since C can only be changed be α, and $\delta(k)[s_2] > \gamma_i(d)$ for $k \geq k^\beta$ by (3) of Procedure 3.2.

Since d is cancelled at $s > s_2$, there is some $\delta(k^{\beta'})$ enumerated in C with $k^{\beta'} < k^\beta$, at some stage $s'(s_2 \leq s' < s)$. Then β' dominates β and setups a shortcut (α, β') at some stage $s''(s_2 \leq s'' < s')$. Hence $\beta'\hat{}g \subseteq TP_{s''}$.

By the definition of the tree, $\beta \not\supseteq \beta'\hat{}g$. Since in addition β is not initialized at s'', $\beta'\hat{}1 \subseteq \beta$. Hence $\beta'\hat{}1$ is accessible at s_1 and $\Gamma^{i'}(C; d^{\beta'})[s_1] \neq 1 = D(d^{\beta'}[s_1])$ (assume β' is an $\mathcal{N}^\alpha_{i'}$-strategy).

But $\Gamma_{i'}(C; d^{\beta'})[s''] = 0 = D(d^{\beta'}[s''])$. Hence $d^{\beta'}[s_1] \neq d^{\beta'}[s'']$. This contradicts with the choice of s_0. □

Next we study some important behaviors of $\mathcal{Q}$-strategies.

Lemma 3.7. *Let τ be a $\mathcal{P}_e$-strategy, $c = \|e\|$ and $\zeta \supseteq \tau\hat{}\infty$ be a $\mathcal{Q}^\tau_{e,n,j}$-strategy.*
(i) Either ζ is initialized infinitely often or a^ζ is eventually fixed;
(ii) If ζ is initialized at most finitely often and accessible infinitely often, then there is a stage s_0 at which both k^ζ and a^ζ are defined and fixed for ever, and for no $k < k^\zeta$ and $s > s_0$, $\theta^\tau(k)[s-1]$ is enumerated in Z^τ;
(iii) Let s_0 be as in (ii) and moreover ζ setups a link (τ, ζ) at stage s_0. Let $s_1 > s_0$ be the earliest τ-expansionary stage. Then (τ, ζ) is travelled at stage s_1, $d^\zeta = d^\zeta[s_0]$ is fixed and either

1. *$\Phi_{n_{c-1}}(Z^\tau; d^\zeta) = 0 \neq 1 = D(d^\zeta)$ and the computation $\Phi_{n_{c-1}}(Z^\tau; d^\zeta)$ is exactly $\Phi_{n_{c-1}}(Z^\tau; d^\zeta)[s_0]$, or*
2. *$M^\tau_{i,j} \cap W_{n_i} \neq \emptyset$ for some $i < c-1$.*

Proof. During the proof we occasionally omit τ and ζ from the superscripts, and we may write X for X_e, etc.. Let $c = \|e\|$.

(i) As in the proof of Lemma 3.6, let s_0 be such that

1. ζ is not initialized after s_0 and k^ζ is defined at s_0 and fixed for ever;
2. For all $k < k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined, and if $k \in K$ then $\Theta(Y, Z; k)[s_0] = 1 = K(k)[s_0]$;
3. For $\mathcal{Q}^\tau$-strategy $\zeta' \subset \zeta$, $a^{\zeta'}$ is defined at s_0 and fixed for ever.

If a is cancelled at $s > s_0$ by σ, then $\Phi(Z; d) = 1 = D(d)$ at s. Suppose d is enumerated in D by τ at $s_2 < s$, then there is a link (τ, ζ) at the beginning of s_2. Suppose the link is setup by ζ at $s_1 < s_2$. As in the proof of Lemma 3.6, we assume $s_1 > s_0$.

Then at s_1, $\Phi(Z; d) = 0 = D(d)$ and $\theta(k^\zeta) > \psi(a)$. The computation $\Phi(Z; d)[s_2]$ is same as that at s_1 by the choice of s_0, and $\theta(k^\zeta)[s_2] > \phi(d)$ by (i)(2) of Procedure 3.5.

Hence at some stage s' between s_2 and s, some $\mathcal{Q}^\tau$-strategy ζ' dominating ζ enumerates $\theta(k^{\zeta'})$ in Z. Then $\zeta'\hat{}\perp \subseteq TP_{s'}$. By the definition of the tree and (ii)(3) of Procedure 3.5, $\zeta'\hat{}1 \subseteq \zeta$. Hence $\zeta'\hat{}1$ is accessible at s_1 and $a^{\zeta'}$ is changed after s_1. This contradicts with the choice of s_0.

(ii) follows immediately from the proof of (i).

(iii) It is obvious that ζ is never initialized after stage s_0. Hence the link is travelled at stage s_1.

By Procedure 3.5, at stage s_0

1. $\Psi(X, Y; a) = 0 \neq 1 = A(a)$;

(2) $\Phi(Z;d) = 0 = D(d)$;
(3) $x \in W_{n_i}$ for $i < c-1$;
(4) $\psi(a) < \theta(k^\zeta)$.

Since s_1 is τ-expansionary, $\Psi(X,Y;a)[s_1] = 1$ and

$$((X,Y) \upharpoonright \psi(a))[s_1] \neq ((X,Y) \upharpoonright \psi(a))[s_0].$$

If $(Y \upharpoonright \psi(a))[s_1] \neq (Y \upharpoonright \psi(a))[s_0]$, then $\Theta(Y,Z;k^\zeta)[s_1-1]$ diverges and $\theta(k')[s_1] > \phi(d)[s_0]$ for $k' \geq k^\zeta$ by (i)(2) of Procedure 3.5. By (ii) the computation $\Phi(Z;d)[s_0]$ no longer changes.

If $(X_{e_i} \upharpoonright \psi(a))[s_1] \neq (X \upharpoonright \psi(a))[s_0]$ for some $i < c-1$ then $x \in M_{i_0,j}$ for i_0 being the greatest such i. □

Let $TP = \liminf_s TP_s$, the next lemma states that TP is infinite and every strategy on TP is eligible to win.

Lemma 3.8. *For every m*
(i) $|TP| \geq m$;
(ii) $TP \upharpoonright m$ is initialized at most finitely often;
(iii) $TP \upharpoonright m$ is accessible infinitely often.

Proof. It is trivial for $m = 0$.

Assume (i)(ii) and (iii) hold for m. Let $\xi = TP \upharpoonright m$. Assume ξ is accessible at $s_0 > m$ and never initialized after s_0.

Case 1, ξ is some $\mathcal{M}$- or $\mathcal{R}$-strategy. (i)(ii) and (iii) hold trivially.

Case 2, ξ is an $\mathcal{N}_i^\alpha$-strategy where $\alpha = top(\xi)$. By Procedure 3.3, ξ always has outcome when it is accessible. Let o be the $<_\Lambda$-least outcome which ξ has infinitely often, then $\xi\hat{\ }o \subseteq TP$.

Hence we may assume that either TP_s is dominated by $\xi\hat{\ }o$ or $TP_s = \xi\hat{\ }o$ for $s > s_0$. At stage $s > s_0$, if $\xi\hat{\ }o$ is initialized, then the initialization could only be launched by α and $o = 0$. If this happens then ξ setups a shortcut (α, ξ) at some stage $s_1 < s$ and $\xi\hat{\ }g \subseteq TP_{s_1}$. But this could happen at most finitely often by the choice of o.

Case 3, ξ is a $\mathcal{P}_e$-strategy. If there are at most finitely many ξ-expansionary stages, then $\xi\hat{\ }0 \subseteq TP$.

If there are infinitely many ξ-expansionary stages, suppose $s_1 > s_0$ is ξ-expansionary but $\xi\hat{\ }\infty$ is not accessible at stage s_1. Then there is a link (ξ, ζ) at the beginning of stage s_1, either it is travelled and no new link is setup at stage s_1 by (ii)(3) of Procedure 3.4 and (i) of Procedure 3.5. Let $s_2 > s_1$ be the next ξ-expansionary stage, then $\xi\hat{\ }\infty$ is accessible.

Case 4, ξ is a $\mathcal{Q}^\tau_{n,j}$-strategy where $\tau = top(\xi)$. Let s_0 be as in (ii) of Lemma 3.7.

If $TP_{s_0} = \xi$ then ξ setups a link (τ, ξ) at stage s_0. Let $s_1 > s_0$ be the next τ-expansionary stage, then by (iii) of Lemma 3.7, either $D \neq \Phi_{n_{c-1}}(Z^\tau)$ or $M^\tau_{i,j} \neq \overline{W}_{n_i}$ for some $i < c-1$ is established for ever at stage s_1. Hence whenever ξ is accessible after stage s_1, $\xi\hat{\ }o$ for some $o \in \{1\} \cup \{m_{i,j} : i < c-1\}$ is also accessible. □

Now we are ready to prove the satisfactions of plus cupping requirements.

Lemma 3.9. *Let α be the unique $\mathcal{M}_e$-strategy on TP, and β be the unique $\mathcal{N}_i^\alpha$-strategy on TP.*

(i) C^α is c.e. and Δ^α is consistent;
(ii) If $\beta\hat{\ }g \not\subset TP$, then $\Delta^\alpha(W_e, C^\alpha; k^\beta)$ converges eventually;
(iii) If $\beta\hat{\ }g \subset TP$, then W_e is computable;
(iv) $\mathcal{M}_e$ is eventually satisfied.

Proof. During the proof, we occasionally omit α and β from the superscripts.

(i) By Lemma 3.8, assume α is not initialized after s. Then $C^\alpha = \bigcup_{t>s} C^\alpha[t]$ and $\Delta^\alpha = \bigcup_{t>s} \Delta^\alpha[t]$, and (i) follows from the construction.

(ii) Let $o = TP(|\beta|)$, then $o = 1$ or 0. By Lemma 3.6, $d = d^\beta$ is eventually fixed. If $o = 1$ then $\beta\hat{\ }g \subseteq TP_s$ for at most finitely many stages, otherwise d could not be fixed. If $o = 0$, then $\beta\hat{\ }g \subseteq TP_s$ for at most finitely many stages too by the definition of TP.

Assume for every $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $\Delta(W_e, C; k^{\beta'})$ eventually converges. By the assumption above, let s_0 be such that

(1) $k^\beta = k^\beta[s_0]$;
(2) For $k \le k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and if $k \in K$ then $K(k)[s_0] = 1 = \Delta(W_e, C; k)[s_0]$;
(3) For $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined and fixed for ever;
(4) β is not initialized and opens no gap after s_0.

Let $s_1 > s_0$ be the earliest α-expansionary stage, then $\Delta(W_e, C; k^\beta)[s_1]$ is defined and fixed for ever.

(iii) By the definition of the tree, for $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $\beta'\hat{\ }g \not\subset TP$. By (ii) above, Lemma 3.6 and 3.8, let s_0 be such that

(1) $k^\beta = k^\beta[s_0]$ and $d^\beta = d^\beta[s_0]$ are defined and fixed for ever;
(2) For $k \le k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and if $k \in K$ then $K(k)[s_0] = 1 = \Delta(W_e, C; k)[s_0]$;
(3) For all $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined and fixed for ever;
(4) β and $\beta\hat{\ }g$ are accessible at s_0 and not initialized after s_0.

Let $(s_m : m \in \omega)$ increasingly enumerate all stages such that $s_m \ge s_0$, both β and $\beta\hat{\ }g$ are accessible at s_m. For each m, let t_m be the first α-expansionary stage after s_m, then $t_m \le s_{m+1}$. Hence β open a gap at s_m while α closes this gap at t_m.

If α closes the gap successfully at t_m, by (4) of Procedure 3.2, $D(d)[t_m] = 1$. By the choice of s_0, d is fixed for ever, and β opens no gap after t_m.

Hence α always closes the gap unsuccessfully. Thus

$$W_e[t_m] \upharpoonright \delta(k)[s_m] = (W_e \upharpoonright \delta(k))[s_m] \text{ and } \delta(k)[s_m] < \delta(k)[t_m] = l^\alpha[t_m] \qquad (*).$$

We claim that

$$W_e[s_{m+1}] \upharpoonright \delta(k)[t_m] = (W_e \upharpoonright \delta(k))[t_m] \qquad (**).$$

If $s_{m+1} = t_m$ then $(**)$ holds trivially.

Assume $s_{m+1} > t_m$. If there exists some stage $t(t_m \le t < s_{m+1})$ at which some ζ enumerates a^ζ in A, then ζ subjects to β.

ζ could not be above $\beta\hat{\ }1$ otherwise $\beta\hat{\ }g$ is initialized. If ζ is above $\beta\hat{\ }g$ then $\beta\hat{\ }g$ is accessible at t and $s_{m+1} \le t$ by the definition of s_{m+1}. This contradicts with the choice of t.

Now it could only be the case that $\zeta = \beta\hat{\ }0$ or subjects to $\beta\hat{\ }0$. Thus ζ is initialized by α when the gap opened at s_m is closed. Hence

$$a^\zeta > t_m > \phi_e(l^\alpha)[t_m] = \phi_e(\delta(k))[t_m]$$

and $(**)$ holds since s_{m+1} is α-expansionary.

(iii) follows from $(*)$ and $(**)$.

(iv) By (i)(ii) and the construction, $K = \Delta(W_e, C)$ if $\beta\hat{\ }g \not\subset TP$ for every $\mathcal{N}^\alpha$-strategy β. (iv) follows from this and (iii). □

Finally we prove the satisfactions of $\mathcal{P}$-strategies.

Lemma 3.10. *τ is the unique $\mathcal{P}_e$-strategy on TP, ζ is the unique $\mathcal{Q}^\tau_{n,j}$-strategy on TP and η is the unique $\mathcal{R}^\tau_{i,j}$-strategy on TP. Let $c = \|e\|$.*
(i) $M^\tau_{i',0}, M^\tau_{i',1}$ and Z^τ are c.e. sets ($i' < c-1$), and Θ^τ is a p.r. functional;
(ii) If $\zeta\hat{\ }\perp \not\subset TP$ then $\Theta^\tau(Y_{e_{c-1}}, Z^\tau; k^\zeta)$ converges;
(iii) $\mathcal{Q}^\tau_{n,j}$ is eventually satisfied;
(iv) $\mathcal{R}^\tau_{i,j}$ is eventually satisfied;
(v) $\mathcal{P}_e$ is eventually satisfied.

Proof. During the proof, we occasionally omit τ, ζ and η from the superscripts, and write X for X_e, etc..

(i) follows from an argument similar to that for (i) of Lemma 3.9.

(ii) Let $o = TP(|\zeta|)$. By Lemma 3.7, we may choose s_0 as in the proof for (ii) of Lemma 3.9 such that

1. a^ζ is defined at stage s_0 and fixed for ever;
2. For all $k \leq k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined, and if $k \in K$ then $K(k)[s_0] = 1 = \Theta(Y, Z; k)[s_0]$;
3. For all $k < k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined and fixed for ever;
4. $\zeta\hat{\ }o$ is accessible at s_0 and never initialized after s_0.

If $o \in \{0\} \cup \{w_{i,j} : (i,j) \in \omega \times 2\}$ then $\Theta^\tau(Y, Z; k)$ converges since $\zeta\hat{\ }\perp$ is never accessible after s_0.

If $o \in \{1\} \cup \{m_{i,j} : (i,j) \in \omega \times 2\}$ then $\Theta^\tau(Y, Z; k)$ converges by (iii) of Lemma 3.7 and (i) of Procedure 3.5.

(iii) Let o be as in (ii). Let a denote the final value of a^ζ.

If $o = \perp$ then $\Psi(X, Y; a)$ diverges and $\mathcal{Q}^\tau_{n,j}$ is satisfied trivially.

Otherwise $d = d^\zeta$ and $x = x^\zeta$ are eventually fixed.

If $o = 0$ then $\Phi(Z; d) \neq 0 = D(d)$. If $o = w_{i,j}$ for some $i < c-1$ then $x \in \overline{W}_{n_i} - M_{i,j}$, hence $M_{i,j} \neq \overline{W}_{n_i}$.

If $o = 1$ then $\Phi(Z; d) \neq 1 = D(d)$. If $o = m_{i,j}$ for some $i < c-1$ then $x \in W_{n_i} \cap M_{i,j}$, hence $M_{i,j} \neq \overline{W}_{n_i}$.

This proved the satisfaction of $\mathcal{Q}^\tau_{n,j}$.

(iv) If there are at most finitely many η-expansionary stages, then $\mathcal{R}^\tau_{i,j}$ is satisfied trivially.

Otherwise, assume η is never initialized after s_0 and s_0 is η-expansionary. Let $(s_m : m \in \omega)$ increasingly enumerate all η-expansionary stages $\geq s_0$.

It suffices to prove that

$$|\{x < \phi_j(l^\eta)[s_m] : x \in (M_{i,0} \cup M_{i,1})[s_{m+1} - 1] - (M_{i,0} \cup M_{i,1})[s_m]\}| \leq 1.$$

Assume $s_{m+1} > s_m$ and some ζ enumerate x^ζ in $M_{i,0}$ at some stage $t(s_m \leq t < s_{m+1})$.

If $t > s_m$ then $\zeta = \eta\hat{\ }0$ or subjects to $\eta\hat{\ }0$. Hence ζ is initialized at s_m and $x_\zeta > s_m \geq \phi_j(l^\eta)[s_m]$.

If $t = s_m$ then $TP_{s_m} = \zeta$ and no more strategies act at s_m. This is the only case that a number less than $\phi_j(l^\eta)[s_m]$ enters $M_{i,0} \cup M_{i,1}$.

(v) By (ii) and Procedure 3.4, if $A = \Psi(X, Y)$ then $K = \Theta(Y, Z)$. Now the satisfaction of $\mathcal{P}_e$ follows immediately from (iii) and (iv). □

This end the proof of Theorem 3.1.

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