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Abstract:

Exploring Complexity of Large Update Interior-Point Methods for $P_*(k)$ Linear Complementarity Problems Based on Kernel Function

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Abstract. Interior Point Methods not only are the most effective methods in practice but also have polynomial-time complexity. The large update interior point methods perform in practice much better than the small update methods which have the best known theoretical complexity. In this paper, motivated by the complexity results for linear optimization based on kernel functions, we extend a generic primal dual interior-point algorithm based on a new kernel function to solve $P_*(k)$ linear complementarity problems. We use some elegant and simple tools to get an iteration bound for the complexity of the algorithm.

Keywords: linear complementary problem, primal-dual interior point method, kernel function, proximity function, large update method, polynomial complexity.

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1. Introduction

After the landmark paper of Karmarkar [8], Linear Optimization (LO) revitalized as an active area of research. Lately the Interior Point Methods (IPMs) have shown their powers in solving LO problems and large classes of other optimization problems (see [13,14,15]. IPMs also are the powerful tools to solve some widely used mathematical problems such as

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Semidefinite Optimization (SDO), Second Order Conic Optimization (SOCO) and Linear Complementary Problem (LCP).

LCPs are one of the most important problems that have many applications in mathematical programming and equilibrium problems. Indeed, it is known that by exploiting the first-order optimality conditions of the optimization problem, any differentiable convex quadratic problem can be formulated into a monotone LCP, i.e. $P_*(0)$ LCP, and vice versa [14]. Variational inequality problems have a relatively close connection with LCPs and widely used in the study of equilibrium problems in, e.g., economics, transportation planning and game theory. The reader can find the basic theory, algorithms and applications in [5].

In this paper, we consider the following LCP:

$$\begin{cases} s = Mx + q, \\ xs = 0, \\ x \geq 0,\, s \geq 0, \end{cases} \qquad \text{(LCP)}$$

where $M \in R^{n\times n}$ is a $P_*(k)$ matrix and $q \in R^n$.

The primal-dual IPM for linear programming was first introduced by Kojima et al. in [9] and extended to wider class of problems such as $P_*(0)$ LCP [4]. The path following IPMs follow the central path approximately to get center. Existence of the central path for $P_*(k)$ LCP has been proved by Kojima et al. [10]. They also generalized primal-dual IPMs to $P_*(k)$ LCP and established the same complexity results as LO case. Nowadays a good measure to evaluate a new variant of IPMs is the capability of the method to extend to the $P_*(k)$ LCPs [7].

Most of polynomial-time interior point algorithms for LO use the logarithmic barrier function as a proximity function. Recently, a new variant of feasible IPMs based on Self-Regular (SR) proximity functions was presented by Peng, Roos and Terlaky [12]. Based on SR-proximities, they provided so far the best worst case theoretical complexity, namely $O(\sqrt{n}\log n \log\frac{n}{\varepsilon})$, for large neighborhood feasible IPMs, for the case when the barrier degree of the corresponding SR function is $1+\log n$. Bai et al. [3] proposed a new primal-dual IPM for LO based on a simple kernel function and they get $O(n\log\frac{n}{\varepsilon})$ iteration complexity. He et al. [6] introduced a self-adjusting IPM for LCP based on a logarithmic barrier function and give some numerical results.

Recently, we proposed a new class of non self regular kernel functions for LO and get $O(q\sqrt{n}(\log n)^{1+\frac{1}{q}}\log\frac{n}{\varepsilon})$, with $q \geq 1$, iteration complexity for large update primal dual IPMs [1]. The aim of this paper is to extend the kernel functions of [1] for solving $P_*(k)$ LCPs and get the same iteration complexity i.e. $O(q\sqrt{n}(\log n)^{1+\frac{1}{q}}\log\frac{n}{\varepsilon})$, for $q \geq 1$. Note that although $P_*(k)$ LCPs are generalization of LO problems, we loose the orthogonality of the components of the search direction vectors i.e., d_x and d_s. Therefore the analysis of search direction and step size is a little different from LO case.

Our kernel function is not strongly convex and not self-regular, hence the analysis is simple. Furthermore, our kernel function is not logarithmic barrier. We use the proximity function $\Psi(v)$ based on this kernel function to measure the proximity between the current iterates and the μ-center in the analysis of the algorithm.

The paper is organized as follows: in Section 2, we recall basic concepts and the notion of the central path. We describe the kernel function and its growth properties for $P_*(k)$ LCP in Section 3. The analysis feasible step size and the amount of the decrease in proximity function during an inner iteration are reported in Section 4. Finally, in Section 5 we explore the total number of iterations for our algorithm.

The following notations are used throughout the paper. The nonnegative and positive orthants are denoted by R_n^+ and R_n^{++}, respectively. For $x = (x_1, x_2, \ldots, x_n) \in R^n$; $x_{min} = min\{x_1, x_2, \ldots, x_n\}$ i.e., the minimal component of x. The vector norm is the 2-norm denoted by $\|.\|$. We also denote the all one-vector of length n by e. The diagonal matrix with the diagonal vector x has been denoted by $X = diag(x)$. The index set J is $J = \{1, 2, \cdots, n\}$. For $x, s \in R^n$, xs denotes the coordinate-wise product (Hadamard product) and $x^T s$ denotes the scalar product. We say $f(t) = \theta(g(t))$ if there exist some positive constants ω_1 and ω_2 such that $\omega_1 g(t) \leq f(t) \leq \omega_2 g(t)$ holds for all $t > 0$. Further, $f(t) = O(g(t))$ if there exists a positive constant ω such that $f(t) \leq \omega g(t)$ holds for all $t > 0$.

2 Preliminary

In this section we review the idea underlying the approach of this paper. The $P_*(k)$ Matrix is first introduced by Kojima et al. [10]. Here, we give some definitions about $P_*(k)$ matrix based on [10] which is the generalization of positive semidefinite matrices.

Definition 2.1: Let $k \geq 0$ be a nonnegative number. A matrix $M \in R^{n\times n}$ is called a $P_*(k)$ matrix if

$$(1+4k)\sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_+(x)} x_i(Mx)_i \geq 0,$$

for all $x \in R^n$,where

$$J_+(x)=\{i \in J : x_i(Mx)_i \geq 0\} \text{ and } J\_(x)=\{i \in J : x_i(Mx)_i < 0\}.$$

Note that for $k=0$ the $P_*(0)$ is the class of positive semidefinite matrices. This implies that the class of convex quadratic programs and LO problems are a subclass of $P_*(0)$ LCPs. In what follows, we give some definitions about convexity concepts which will be essential tools in our analysis.

Definition 2.2: A C^2 function $f : X(\subset R) \rightarrow R$ is strongly convex if and only if there exists $m_0 > 0$ such that $f''(x) \geq m_0$.

Definition 2.3: A function $f : X(\subset R) \rightarrow R$ is exponentially convex if and only if

$$f\left(\sqrt{x_1 x_2}\right) \leq \frac{1}{2}\left(f(x_1) + f(x_2)\right).$$

The following results play an important role in determining the search direction, the reader can find proofs of these results in [10].

Proposition 2.4
For $P_*(k)$ matrix $M \in R^{n\times n}$, the matrix

$$M' = \begin{bmatrix} -M & I \\ S & X \end{bmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X, S \in R^{n\times n}$.

The following corollary is used to prove the unique solution for modified Newton-system.

Corollary 2.5
Let $M \in R^{n\times n}$ be a $P_*(k)$ matrix and $x, s \in R^n_{++}$. Then, for all $a \in R^n$ the system

$$\begin{cases} -M\Delta x + \Delta s = 0 \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

The basic idea of primal-dual IPMs is to replace the second equation in system (LCP), *the complementary condition*, by the parameterized equation $xs = \mu e$, with $\mu > 0$. Parameter μ is referred to as the *central path parameter*. This replacement leads us to have the following system:

$$\begin{cases} s = Mx + q, \\ xs = \mu e, \\ x > 0,\, s > 0, \end{cases} \tag{2.1}$$

where $\mu > 0$. Without loss of generality, we assume that (LCP) is strictly feasible, i.e., there exists (x_0, s_0) such that $s_0 = Mx_0 + q, x_0 > 0$, $s_0 > 0$ with $\Psi(x_0, s_0, \mu_0) \le \tau$ for $\mu^0 > 0$. Indeed, we may not have an available strictly feasible point(x_0, s_0). In order to solve this difficulty, we embed (LCP) to an artificial LCP which has a strictly feasible point [10]. For given strictly feasible point (x_0, s_0), we can always find a $\mu^0 > 0$ such that $\Psi(x_0, s_0, \mu_0) \le \tau$. Since M is a $P_*(k)$ matrix and (LCP) is strictly feasible, then (2.1) has a unique solution for any $\mu > 0$. We denote the solution of (2.1) as $(x(\mu), s(\mu))$, the μ-center, for given $\mu > 0$. The set of unique solutions $\{(x(\mu), s(\mu)) : \mu > 0\}$ of system (2.1) forms the so-called *central path* of (LCP). As $\mu \to 0$, the sequence $(x(\mu), s(\mu))$ approaches to the solution (x, s) of the (LCP) [10]. By introducing the notations:

$$d = \sqrt{\frac{x}{s}}\ ,\ v = \sqrt{\frac{xs}{\mu}}\ ,\ d_x = \frac{v\Delta x}{x}\ ,\ d_s = \frac{v\Delta s}{s} \tag{2.2}$$

we can rewrite the system (2.1) as the following scaled Newton-system:

$$\begin{cases} -\overline{M}dx + ds = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.3}$$

where $\overline{M} = DMD$ and $D = diag(d)$. The second equation in (2.3) is called *the scaled centering equation.*

Now we consider a strictly convex function $\Psi(v)$ which is minimal at $v = e$ and $\Psi(e) = 0$. We replace the scaled centering equation in (2.3) by

$$d_x + d_s = -\nabla\Psi(v).$$

Therefore, we can get the following modified Newton system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v\nabla\Psi(v), \end{cases} \tag{2.4}$$

Since M is a $P_*(k)$ matrix and (LCP) is strictly feasible, the system (2.4) uniquely defines a search direction $(\Delta x, \Delta s)$ using Corollary 2.5. Throughout the paper we assume that a threshold parameter τ and a fixed barrier update parameter θ are given. We also have $\tau = O(n)$ and $0 < \theta < 1$.

The algorithm works as follows: starting with a strictly feasible point (x,s) in a τ-neighborhood of the current μ-center, we decrease μ to $\mu_+ = (1-\theta)\mu$ for some fixed $\theta \in (0,1)$ and then we solve the modified Newton-system (2.4) to obtain the unique search direction. The positiveness condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. By repeating this process, we find a new iterate (x_+, s_+) in a τ-neighborhood of the μ_+-center. Now, we let $\mu = \mu_+$ and $(x,s) := (x_+, s_+)$. Again, μ is reduced by the factor $1-\theta$ and the modified Newton system has been solved targeting at the new μ_+-center, and so on. This procedure is repeated until μ is small enough, say until $n\mu \le \varepsilon$. Using the proximity function $\Psi(v)$, in order to find the search direction and to measure the proximity between the current iterates and the μ-center, the generic primal-dual algorithm for $P_*(k)$ LCPs [13] can be described in Algorithm 1.

Algorithm1. Generic Primal-Dual Algorithm

Input:
a proximity function $\Psi(v)$;
a threshold parameter $\tau > 0$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;

begin
 $x := x_0;\ s := s_0;\ \mu := \mu_0;$
 while $n\mu \ge \varepsilon$ **do**
 begin
  $\mu := (1-\theta)\mu;$
  $v = \sqrt{\dfrac{xs}{\mu}};$
  while $\Psi(v) \ge \tau$ **do**
  begin
   $x := x + \alpha\Delta x;$
   $s := s + \alpha\Delta s;$
  end
 end
end

In Algorithm 1, the inner “while loop” is called inner iteration and the outer “while loop” is called outer iteration. Each outer iteration consists of an update of parameter μ and a sequence of (one or more) inner iterations. The total number of inner iterations is referred to as iteration complexity of the algorithm. Usually this number is described as a function of the dimension n and ε.

Remark 2.6
The choice of θ, the so-called *barrier update parameter*, plays an important role in theory and practice of IPMs. Usually, if θ is a constant independent of the problem dimension n, for instance $\theta = \frac{1}{2}$, then we call the algorithm a *large-update* (or *long-step*) method. If θ depends on the dimension of the problem, such as $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is named a *small-update* (or *short-step*) method. Recall that small-update methods have the best iteration bound. On the other hand, large-update methods that are in practice much more efficient than small-update methods [13], have a worst-case iteration bound. This phenomenon is what we called ”The gap between theory and practice”.

Remark 2.7
Up till, only algorithms based on the logarithmic barrier functions and SR functions were considered for the analysis of Algorithm 1.

3 The kernel function and growth behavior

In [1], we introduced the kernel function

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{1}{q}\left(e^{t^{-q}-1} - 1\right), \qquad (3.1)$$

for $t > 0$ and $q \geq 1$, and the proximity function induced by the kernel function as:

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i) \qquad (3.2)$$

to analysis Algorithm 1 for LO. The complexity result for large-update LO based on the kernel function (3.1) is [1]:

$$O\left(q\sqrt{n}(\log n)^{1+\frac{1}{q}} \log\frac{n}{\varepsilon}\right). \qquad (3.3)$$

In this section, we investigate and recall some properties of (3.1), which determines the search direction in the Algorithm 1. We also verify the growth behavior of the kernel function (3.1).

In the analysis of Algorithm 1, we need its first three derivatives with respect to *t*. For ease of reference, we give them here. We have,

$$\psi'(t) = t - \frac{e^{t^{-q}-1}}{t^{q+1}}, \qquad (3.4)$$

$$\psi''(t) = 1 + \frac{e^{t^{-q}-1}\left(q + qt^{q} + t^{q}\right)}{t^{2q+2}} \geq 1, \qquad (\text{for } t > 0). \qquad (3.5)$$

$$\psi'''(t) = \frac{e^{t^{-q}-1}\left(-q^{2} + \left(1-q-2q^{2}\right)t^{q} - \left(2+3q+q^{2}\right)t^{2q}\right)}{t^{3q+3}}, \quad (\text{for } t > 0). \quad (3.6)$$

It follows that $\psi(1) = \psi'(1) = 0$ and $\psi''(t) \geq 1$, which show that $\psi(t)$ is indeed a kernel function. It is also quite straightforward that

$$\lim_{t \to 0} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty .$$

We proceed by describing some recent results. Due to conditions $\psi(1) = \psi'(1) = 0$, we can completely describe $\psi(t)$ by its second derivative as follows:

$$\psi(t) = \int_{1}^{t} \int_{1}^{\xi} \psi''(\varsigma) d\varsigma \, d\xi . \qquad (3.7)$$

In the analysis of the algorithm, we also use the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) := \frac{1}{2}\left\|\nabla \Psi(v)\right\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\left(\psi'(v_i)\right)^{2}} = \frac{1}{2}\left\|d_x + d_s\right\| . \qquad (3.8)$$

Note that $\Psi(v)$ is strictly convex and has a minimal at $v = e$ where as the minimal value is zero. So we have

$$\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e .$$

From aforesaid properties, we have the following results.

Lemma 3.1 *$\psi''(t)$ is monotonically decreasing for all $t > 0$ and $q \geq 1$.*

Proof: Since for all $q \geq 1$, we have

$$\begin{cases} \Delta = \left(1-q-2q^{2}\right)^{2} - 4q^{2}\left(2+3q+q^{2}\right) = -8q^{3} - 13q^{2} + 1 < 0 \\ -\left(2+3q+q^{2}\right) < 0 \end{cases}$$

So, for all $t > 0$,

$$-q^{2} + \left(1-q-2q^{2}\right)t^{q} - \left(2+3q+q^{2}\right)t^{2q} < 0$$

and from (3.6), we have

$$\psi'''(t) < 0, \qquad (\text{for all } t > 0).$$

Therefore, $\psi''(t)$ is monotonically decreasing.

Lemma 3.2 $\psi(t)$ *defined as (3.1) is exponentially convex. i.e.*

$$\psi(\sqrt{t_1 t_2}) \le \frac{1}{2}[\psi(t_1)+\psi(t_2)] \qquad ,\forall t_1, t_2 > 0.$$

Proof: see [1].

Lemma 3.3 *We have*

$a)$ $\frac{1}{2}(t-1)^2 \le \psi(t) \le \frac{1}{2}\psi'(t)^2,$ for $t > 0$.

$b)$ $\Psi(v) \le 2\delta(v)^2$

$c)$ $\|v\| \le \sqrt{n} + \sqrt{2\Psi(v)} \le \sqrt{n} + 2\delta(v).$

Proof: see [2].

Note that at the start of any outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \le \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during the process of the algorithm the largest values of $\Psi(v)$ occur just after the updates of μ. Now, we investigate the growth behavior of $\Psi(v)$ after μ-update.

Lemma 3.4 *Let* $\beta \ge 1$. *Then*

$$\psi(\beta t) \le \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2. \tag{3.9}$$

Proof: Using the definition of $\psi(t)$ we may write

$$\begin{aligned}\psi(\beta t) - \psi(t) &= \frac{(\beta t)^2 - 1}{2} + \frac{1}{q}\left(e^{(\beta t)^{-q}-1} - 1\right) - \psi(t)\\ &= \frac{(\beta t)^2 - 1}{2} - \frac{t^2 - 1}{2} + \frac{1}{q}\left(e^{(\beta t)^{-q}-1} - 1\right) - \frac{1}{q}\left(e^{t^{-q}-1} - 1\right)\\ &= \frac{(\beta t)^2 - t^2}{2} + \frac{1}{eq}\left(e^{(\beta t)^{-q}} - e^{t^{-q}}\right)\\ &\le \frac{1}{2}(\beta^2 - 1)t^2\end{aligned}$$

The last inequality follows from the relation $e^{(\beta t)^{-q}} - e^{t^{-q}} \le 0$, for $\beta \ge 1$. Thus, the lemma follows. ■

Corollary 3.1 *Let* $0 \le \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$. *Then*

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

Proof: Using Lemma 3.4, with $\beta = \frac{1}{\sqrt{1-\theta}}$, we may write

$$\Psi(v_+) = \Psi\left(\frac{v}{\sqrt{1-\theta}}\right) = \sum_{i=1}^{n} \psi(\beta v_i) \le \sum_{i=1}^{n} \left(\psi(v_i) + \frac{1}{2}(\beta^2 - 1)v_i^2\right) = \Psi(v) + \frac{\theta \|v\|^2}{2(1-\theta)}.$$

Using Lemma 3.3, we obtain

$$\Psi(v_+) \le \Psi(v) + \frac{\theta\left(\sqrt{n} + \sqrt{2\Psi(v)}\right)^2}{2(1-\theta)} \le \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

This completes the proof. ■

4 A Default Value for Step Size

In this section we evaluate the feasible step size α in which the proximity function is decreasing. We also derive an upper bound for the decrease. Note that although $P_*(k)$ LCPs are generalization of LO problems, we loose the orthogonality of the vectors d_x and d_s in this case. Therefore, the analysis of step size is different from LO case.

After a damped step, with step size α, we have

$$x_+ = x + \alpha\Delta x \quad ; \quad s_+ = s + \alpha\Delta s .$$

Using (2.2), we get

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha\, d_x),$$

and

$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha\, d_s).$$

Thus, we obtain

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha\, d_x)(v + \alpha\, d_s). \qquad (4.1)$$

Hence, using e-convexity of ψ, we have

$$\Psi(v_+) = \Psi\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) \le \frac{1}{2}\left[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right] .$$

Defining

$$f(\alpha) = \Psi(v_+) - \Psi(v),$$

and

$$f_1(\alpha)=\frac{1}{2}\left[\Psi(v+\alpha d_x)+\Psi(v+\alpha d_s)\right]-\Psi(v),$$

we have $f(\alpha)\le f_1(\alpha)$ and $f_1(\alpha)$ is a convex function, since Ψ is convex and in the first two terms of right hand side, the argument of Ψ is linear with respect to α. Obviously,

$$f(0)=f_1(0)=0.$$

By taking the first derivative with respect to α, we get

$$f_1'(\alpha)=\frac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i+\alpha d_{x_i})d_{x_i}+\psi'(v_i+\alpha d_{s_i})d_{s_i}\right).$$

Using (3.8), we have

$$f_1'(0)=\frac{1}{2}\nabla\Psi(v)^T(d_x+d_s)=-\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v)=-2\delta(v)^2 \qquad (4.2)$$

The second derivative of $f_1(\alpha)$ with respect to α is

$$f_1''(\alpha)=\frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i+\alpha d_{x_i})d_{x_i}{}^2+\psi''(v_i+\alpha d_{s_i})d_{s_i}{}^2\right). \qquad (4.3)$$

Now, since M is a $P_*(k)$ matrix and from (2.4) have $M\Delta x=\Delta s$. thus, for $\Delta x,\Delta s\in R^n$, we have

$$\left(1+4k\right)\sum_{i\in J_+(x)}\Delta x_i\Delta s_i+\sum_{i\in J_-(x)}\Delta x_i\Delta s_i\ge 0,$$

where $J_+=\{i\in J:\Delta x_i\Delta s_i\ge 0\}$ and $J_-=J-J_+$. Since

$$d_x\,d_s=\frac{v^2\Delta x\Delta s}{xs}=\frac{\Delta x\Delta s}{\mu}$$

and $\mu>0$, thus

$$\left(1+4k\right)\sum_{i\in J_+(x)}d_{x_i}d_{s_i}+\sum_{i\in J_-(x)}d_{x_i}d_{s_i}\ge 0, \qquad (4.4)$$

For notational convenience, we define the following notations:

$$\delta:=\delta(v),\ \sigma_+=\sum_{i\in J_+}d_{x_i}d_{s_i}\ ,\ \sigma_-=-\sum_{i\in J_-}d_{x_i}d_{s_i}$$

To estimate the bound for $\|d_x\|$ and $\|d_s\|$, we need the following technical lemma.

Lemma 4.1 *We have* $\sigma_+\le\delta^2$ *and* $\sigma_-\le(1+4k)\delta^2$.

Proof: From the definition of σ_+ and σ_-, we have

$$\sigma_+=\sum_{i\in J_+}d_{x_i}d_{s_i}\le\frac{1}{4}\sum_{i\in J_+}\left(d_{x_i}+d_{s_i}\right)^2\le\frac{1}{4}\sum_{i=1}^{n}\left(d_{x_i}+d_{s_i}\right)^2=\delta^2$$

From (4.4), we also have

$$(1+4k)\sigma_+-\sigma_-\ge 0.$$

Thus

$$\sigma_- \le (1+4k)\sigma_+ \le (1+4k)\delta^2 .$$

This completes the lemma.■

In the following lemma, we estimate an upper bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 4.2 *We have*

$$\sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right) \le 4(1+2k)\delta^2 \ , \ \|d_x\| \le 2\sqrt{1+2k}\,\delta \quad , \ \|d_s\| \le 2\sqrt{1+2k}\,\delta .$$

Proof: Since $\delta = \frac{1}{2}\|d_x + d_s\|$ and $\sum_{i\in J} d_{x_i} d_{s_i} = \sigma_+ - \sigma_-$,

$$2\delta = \|d_x + d_s\| = \sqrt{\sum_{i=1}^{n}\left(d_{x_i} + d_{s_i}\right)^2} = \sqrt{\sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right) + 2(\sigma_+ - \sigma_-)}$$

Using (4.4), we have

$$2\delta \ge \sqrt{\sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right) + 2\left(\frac{1}{1+4k}\sigma_- - \sigma_-\right)} = \sqrt{\sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right) - \frac{8k}{1+4k}\sigma_-} \ .$$

Hence,

$$4\delta^2 \ge \sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right) - \frac{8k}{1+4k}\sigma_- .$$

Using Lemma 4.1, we obtain

$$4(1+2k)\delta^2 \ge 4\delta^2 + \frac{8k}{1+4k}\sigma_- \ge \sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right).$$

Thus, we have

$$2\sqrt{(1+2k)}\delta \ge \sqrt{\sum_{i=1}^{n}\left(d_{x_i}{}^2 + d_{s_i}{}^2\right)} \ge \|d_x\| .$$

Similarly, we can obtain $2\sqrt{(1+2k)}\delta \ge \|d_s\|$. This completes the proof. ■

To estimate an upper bound for the decrease of the proximity function during an iteration, we need the following technical lemmas.

Lemma 4.3 *We have*

$$f_1''(\alpha) \le 2(1+2k)\delta^2 \psi''(v_{\min} - 2\alpha\sqrt{1+2k}\delta) . \qquad (4.5)$$

Proof: Using Lemma 4.2, since $\|d_x\| \le 2\sqrt{(1+2k)}\delta$ and $\|d_s\| \le 2\sqrt{(1+2k)}\delta$, thus $v_i + \alpha dx_i \ge v_{min} - 2\alpha\sqrt{1+2k}\delta$ and $v_i + \alpha ds_i \ge v_{min} - 2\alpha\sqrt{1+2k}\delta$. Now, using (4.3) and Lemma 3.1, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i + \alpha\, d_{x_i})\, d_{x_i}{}^2 + \psi''(v_i + \alpha\, d_{s_i})\, d_{s_i}{}^2\right)$$

$$\le \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_{\min} - 2\alpha\sqrt{1+2k}\delta) d_{x_i}{}^2 + \psi''(v_{\min} - 2\alpha\sqrt{1+2k}\delta) d_{s_i}{}^2\right)$$

$$\leq 2(1+2k)\delta^2\psi''\left(v_{\min}-2\alpha\sqrt{1+2k}\delta\right).$$

This completes the lemma.■

Since $f_1(\alpha)$ is convex, we will have $f_1'(\alpha)\leq 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal and vice versa. Due to this fact, we establish the following lemma which is important to define a default value for step size.

Lemma 4.4 *We have* $f_1'(\alpha)\leq 0$ *if* α *satisfies the following inequality*

$$-\psi'(v_{\min}-2\alpha\delta\sqrt{1+2k})+\psi'(v_{\min})\leq\frac{2\delta}{\sqrt{1+2k}}. \qquad (4.6)$$

Proof: We know that

$$f_1'(\alpha)=f_1'(0)+\int_0^\alpha f_1''(\zeta)d\zeta.$$

Using (4.2) and Lemma 4.3, we have

$$\begin{aligned}
f_1'(\alpha)&\leq -2\delta^2+2(1+2k)\delta^2\int_0^\alpha \psi''(v_{\min}-2\zeta\sqrt{1+2k}\delta)d\zeta\\
&=-2\delta^2-\sqrt{1+2k}\,\delta\int_0^\alpha \psi''(v_{\min}-2\zeta\sqrt{1+2k}\delta)d(v_{\min}-2\zeta\sqrt{1+2k}\,\delta)\\
&=-2\delta^2-\sqrt{1+2k}\,\delta\left(\psi'(v_{\min}-2\alpha\delta\sqrt{1+2k})-\psi'(v_{\min})\right)\\
&\leq -2\delta^2+\sqrt{1+2k}\,\delta\frac{2\delta}{\sqrt{1+2k}}=0,
\end{aligned}$$

where the last inequality obtains from the assumption. ■

We note that $\psi''(t)\geq 1$ so the function $-\frac{1}{2}\psi'(t)$ has the inverse function. Suppose that $\rho:[0,\infty)\to(0,1]$ is the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0,1]$. We want to solve inequality (4.6) for the largest possible α, irrespective to the value of v_{min}. Since $\psi''(t)$ is decreasing, the derivative of the expression at the left in (4.6) to v_{min} is negative. Therefore, with δ fixed, in order to hold equality in (4.6), we must find the minimum value of $\psi'(v_{\min})$ and the maximum value of $\psi'(v_{\min}-2\alpha\delta\sqrt{1+2k})$. We have

$$\delta=\frac{1}{2}\|\nabla\Psi(v)\|\geq\frac{1}{2}|\psi'(v_{min})|\geq-\frac{1}{2}\psi'(v_{min}),$$

and equality holds if and only if v_{min} is the only coordinate in v that differs from one, and $v_{min}\leq 1$. Hence, the worst situation for the step size occurs when v_{min} satisfies

$$-\frac{1}{2}\psi'(v_{min})=\delta \qquad (4.7)$$

In that case $\psi'(v_{min})$ has the minimum value. Now, the inequality (4.6) reduces to

$$-\frac{1}{2}\psi'(v_{min}-2\alpha\delta\sqrt{1+2k})\leq\frac{2\delta}{\sqrt{1+2k}}$$

The derivative with respect to α in the left side of this inequality is positive. So, the largest possible value of α satisfying above inequality will satisfy

$$-\frac{1}{2}\psi'(v_{min}-2\alpha\delta\sqrt{1+2k})=\frac{2\delta}{\sqrt{1+2k}} \qquad (4.8)$$

Due to the definition of ρ, (4.7) and (4.8) can be rewritten as:

$$v_{\min}=\rho(\delta),$$

and

$$v_{\min}-2\alpha\delta\sqrt{1+2k}=\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right).$$

This implies,

$$\overline{\alpha}=\frac{1}{2\delta\sqrt{1+2k}}\left[\rho(\delta)-\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)\right]=\frac{1}{2\delta\sqrt{1+2k}}\int_{\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta}^{\delta}\rho'(\sigma)\,d\sigma \qquad (4.9)$$

In the worst case situation, the step size $\overline{\alpha}$ is the largest possible solution for (4.6).

Now, we are going to introduce a default value for step size in the algorithm. Using the definition of ρ, we have

$$-\psi'(\rho(\delta))=2\delta$$

Taking the derivative with respect to δ, we obtain

$$\rho'(\delta)=\frac{-2}{\psi''(\rho(\delta))}<0$$

Thus, from (4.9), we can get upper and lower bounds for $\overline{\alpha}$ as follow:

$$\frac{1}{(1+2k)\psi''\left(\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)\right)}\leq\overline{\alpha}\leq\frac{1}{(1+2k)\psi''(\rho(\delta))}.$$

In the sequel, we use the following $\tilde{\alpha}$ as our default value for the step size,

$$\tilde{\alpha}=\frac{1}{(1+2k)\psi''\left(\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)\right)}. \qquad (4.10)$$

Note that $\tilde{\alpha}\leq\overline{\alpha}$.

5 Decrease of the Proximity and Complexity

In this section, we first obtain an estimate for the value of $f(\tilde{\alpha})$, then, by using some technical lemmas, we conclude the iteration complexity.

5.1 Estimate Value of $f(\tilde{\alpha})$

To get an estimate value for $f(\tilde{\alpha})$, we need the following technical lemma where its elementary proof can be found in [12].

Lemma 5.1 *Let* $h(t)$ *be a twice differentiable convex function with* $h(0)=0$ *,* $h'(0)<0$ *and which attains its (global) minimum at* $t^*>0$. *If* $h''(t)$ *is increasing for* $t\in[0,t^*]$, *then*

$$h(t)\le\frac{th'(0)}{2},\qquad 0\le t\le t^*. \tag{5.1}$$

Now, we can present an upper bound for $f(\tilde{\alpha})$ as follows:

Lemma 5.2 *If the step size* α *is such that* $\alpha\le\bar{\alpha}$ *, then*

$$f(\alpha)\le-\alpha\delta^2. \tag{5.2}$$

Proof: Let the univariate function h be such that

$$h(0)=f_1(0)=0,\quad h'(0)=f_1'(0)=-2\delta^2,$$

and

$$h''(\alpha)=2(1+2k)\delta^2\,\psi''(v_{\min}-2\alpha\delta\sqrt{1+2k}).$$

Using Lemma 4.3, we have $f_1''(\alpha)\le h''(\alpha)$ and as a consequence $f_1'(\alpha)\le h'(\alpha)$ and $f_1(\alpha)\le h(\alpha)$. We may write

$$\begin{aligned} h'(\alpha) &= h'(0)+\int_0^\alpha h''(\xi)d\xi \\ &= -2\delta^2+2(1+2k)\delta^2\int_0^\alpha \psi''(v_{min}-2\xi\sqrt{1+2k}\delta)d\xi \\ &= -2\delta^2-\sqrt{1+2k}\delta\left(\psi'(v_{min}-2\alpha\sqrt{1+2k}\delta)-\psi'(v_{min})\right) \end{aligned}$$

Therefore, from Lemma 4.4, we can get

$$h'(\alpha)\le-2\delta^2+\sqrt{1+2k}\delta\frac{2\delta}{\sqrt{1+2k}}=0,$$

for $\alpha\le\bar{\alpha}$. On the other hand, since $\psi''(t)$ is decreasing, $h''(\alpha)$ is increasing with respect to α. Hence, by applying Lemma 5.1, we obtain

$$f_1(\alpha)\le h(\alpha)\le\frac{1}{2}\alpha h'(0)=-\alpha\delta^2.$$

This completes the proof, since $f(\alpha)\le f_1(\alpha)$. ■

Corollary 5.3 *Let* ρ *be the inverse function of the restriction of* $-\frac{1}{2}\psi'(t)$ *to the interval* $(0,1]$ *and* $\tilde{\alpha}$ *be defined as in (4.10). Then*

$$f(\tilde{\alpha}) = -\frac{\delta^2}{(1+2k)\psi''\left(\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)\right)}. \qquad (5.3)$$

Proof: The corollary follows immediately if we apply Lemma 5.2 to the default step size $\tilde{\alpha}$.

Now, we apply the so far obtained results to our proximity function. To this end, we need to compute $\rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)$. Letting $s = \rho\left(\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta\right)$, we have $-\psi'(s) = 2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta$, from the definition of ρ. So, from (3.4) ,(3.5) and (4.10), we may write

$$\tilde{\alpha} = \frac{1}{(1+2k)\psi''(s)} = \frac{1}{(1+2k)\left(1+\frac{e^{s^{-q}-1}\left(q+qs^q+s^q\right)}{s^{2q+2}}\right)}, \qquad (5.4)$$

and

$$e^{s^{-q}-1} = s^{q+1}\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta + s\right).$$

Thus,

$$s^{-q} = 1+(q+1)\log s + \log\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta + s\right).$$

Note that $0 \le s \le 1$. Hence

$$s^{-q} \le 1+\log\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta + 1\right),$$

and

$$\frac{1}{s^{q+1}} \le \left(1+\log\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta + 1\right)\right)^{1+\frac{1}{q}}.$$

Therefore, we conclude that

$$\tilde{\alpha} \geq \frac{1}{(2q+1)(1+2k)\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta+1\right)\left(1+\log\left(2\left(1+\frac{1}{\sqrt{1+2k}}\right)\delta+1\right)\right)^{1+\frac{1}{q}}}$$

$$\geq \frac{1}{(4\delta+1)(2q+1)(1+\log(4\delta+1))^{1+\frac{1}{q}}} = \Theta\left(\frac{1}{8q\delta(\log(4\delta+1))^{1+\frac{1}{q}}}\right),$$

where the last inequality obtains from $\frac{1}{\sqrt{1+2k}} \leq 1$. Now, using Lemma 5.2, we have

$$f(\tilde{\alpha}) = \Theta\left(-\frac{\delta}{8q(\log(4\delta+1))^{1+\frac{1}{q}}}\right). \qquad (5.5) ■$$

5.2 Iteration complexity

After the update of μ to $(1-\theta)\mu$, we have, by Corollary 3.1,

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

At the start of an outer iteration we have $\Psi(v) \leq \tau$. We will assume that $\tau = O(n)$, and for the purpose of analyzing the large-update method, we have $\theta = \Theta(1)$. We need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after μ-update. We denote the value of $\Psi(v)$ after the μ-update as Ψ_0, and the subsequent values are denoted as Ψ_k, $k = 1,2,\ldots,K$, with K denoting the total number of inner iterations in an outer iteration. By Corollary 3.1,

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2n\tau} + n\right) = O\left(n + \frac{n\theta}{1-\theta}\right) = O(n). \qquad (5.6)$$

The decrease on each inner iteration is given by (5.5), i.e.,

$$\Psi_{k+1} \leq \Psi_k - \bar{\kappa}\frac{\delta}{q(\log(4\delta+1))^{1+\frac{1}{q}}}, \qquad k = 0,1,\ldots,K-1,$$

where $\bar{\kappa}$ is some positive constant. Since the decrease depends monotonically on δ, and from Lemma 3.3, we have

$$\delta = \delta(v) \geq \sqrt{\frac{\Psi(v)}{2}},$$

therefore, by assuming

$$1 \leq \tau \leq \Psi_{k+1} \leq \Psi_k, \qquad k = 0,1,\ldots,K-1,$$

we may express the decrease in terms of Ψ. Thus, from the aforementioned inequalities, we obtain

$$\Psi_{k+1} \le \Psi_k - \kappa \Delta\Psi_k, \qquad k = 0,1,\ldots,K-1, \qquad (5.7)$$

where $\Delta\Psi_k$ is the function defined as below

$$\Delta\Psi_k = \Theta\left(\frac{\Psi_k^{\frac{1}{2}}}{q(\log \Psi_k)^{1+\frac{1}{q}}} \right),$$

and κ is some positive constant. We need the following technical result to get the iteration bound.

Lemma 5.4 *If* $\alpha \in [0,1]$, *then*

$$(1+t)^{\alpha} \le 1 + \alpha t, \qquad (\forall\, t \ge -1). \qquad (5.8)$$

Proof: see [11].

We use the following lemma to get the number of inner iterations and therefore the total number of iterations for Algorithm 1.

Lemma 5.5 *Considering (5.7), we have*

$$K = O\left(\frac{\Psi_0}{\frac{1}{2}\kappa\Psi_0} \right).$$

Proof: We may write $\Delta\Psi_k$ as

$$\Delta\Psi_k = f(\Psi_k)\Psi_k^{\frac{1}{2}}, \qquad (5.9)$$

where $f(\Psi) = \Theta\left(\frac{1}{q(\log \Psi)^{1+\frac{1}{q}}} \right)$ is monotonically decreasing. Substitution on (5.7) gives

$$\Psi_{k+1} \le \Psi_k - \kappa f(\Psi_k)\Psi_k^{\frac{1}{2}}, \qquad k = 0,1,\ldots,K-1.$$

From this inequality, we derive

$$0 \le \Psi_{k+1}^{\frac{1}{2}} \le \left(\Psi_k - \kappa f(\Psi_k)\Psi_k^{\frac{1}{2}} \right)^{\frac{1}{2}} = \Psi_k^{\frac{1}{2}} \left(1 - \kappa f(\Psi_k)\Psi_k^{-\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\le \Psi_k^{\frac{1}{2}} \left(1 - \frac{1}{2}\kappa f(\Psi_k)\frac{1}{2}\Psi_k^{-\frac{1}{2}} \right) = \Psi_k^{\frac{1}{2}} - \frac{1}{2}\kappa f(\Psi_k),$$

where the last inequality obtains from Lemma 5.4. Since $f(\Psi)$ is monotonically decreasing, we have

$$0 \le \Psi_k^{\frac{1}{2}} - \frac{1}{2}\kappa f(\Psi_k) \le \Psi_k^{\frac{1}{2}} - \frac{1}{2}\kappa f(\Psi_0), \qquad k = 0,1,\ldots,K-1 .$$

This implies

$$\Psi_k^{\frac{1}{2}} \le \Psi_0^{\frac{1}{2}} - \frac{1}{2}k\kappa f(\Psi_0), \qquad k = 0,1,\ldots,K .$$

Taking $k = K$, we obtain

$$0 \le \Psi_0^{\frac{1}{2}} - \frac{1}{2}K\kappa f(\Psi_0),$$

which implies

$$K \le \frac{\Psi_0^{\frac{1}{2}}}{\frac{1}{2}\kappa f(\Psi_0)} = \frac{\Psi_0}{\frac{1}{2}\kappa\Delta\Psi_0} .$$

The equality is due to (5.9). This proves the lemma. ■

Using (5.6), we have $\Psi_0 = O(n)$. Thus, from Lemma 5.5, we obtain that the number of inner iterations is bounded above as follows

$$K \le \left\lfloor \frac{\Psi_0^{\frac{1}{2}}}{\frac{1}{2}\kappa f(\Psi_0)} \right\rfloor = \left\lfloor O\left(q\sqrt{n}(log_e n)^{1+\frac{1}{q}} \right) \right\rfloor .$$

The iteration complexity of the algorithm is obtained by multiplying this number by the number of outer iteration, which is bounded above by $O\left(log_e^{\frac{n}{\varepsilon}} \right)$, see [13]. Omitting the integer brackets, which does not change the order, the iteration complexity is given by

$$O\left(q\sqrt{n}\left(log_e^{n}\right)^{1+\frac{1}{q}} log_e^{\frac{n}{\varepsilon}} \right) .$$

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References

[1] K. Amini and M.R. Peyghami. An Interior-Point Algorithm for Linear Optimization Based on a New Kernel Function, *Southeast Asian Bulletin of Mathematics*, 29, pp. 651-667, 2005.

[2] Y.Q. Bai, M. El ghami and C. Roos. A new efficient large-update primal-dual interior-point method based on a finite barrier, *SIAM Journal on Optimization*, 13(3), pp. 766-782 (electronic), 2003.

[3] Y.Q. Bai and C. Roos. A polynomial-time algorithm for linear optimization based on a new simple kernel function, *Optimization Methods and Software*, 18(6), pp. 631-646, 2003.

[4] G.M. Cho. Log-barrier method for two-stage quadratic stochastic programming, *Applied Mathematics and Computation*, 164, pp. 45-69, 2005.

[5] R.W. Cottle, J.S. Pang, and R.E. Stone. *The Linear Complementarity Problem*, Academic Press Inc., San Diego, CA, 1992.

[6] S. He, X. Li, and S. Pan. A self-adjusting interior point algorithm for linear complementarity problems, *Computers and Mathematics with Applications*, 50, pp. 33-40, 2005.

[7] T. Ills and M. Nagy. The Mizuno-Todd-Ye predictor-corrector algorithm for sufficient matrix linear complementarity problem, *Technical Report*, Eotvos Lorand University of Sciences, 2004.

[8] N. K. Karmarkar. A new polynomial-time algorithm for linear programming, *Combinatorica*, 4, pp. 373-395, 1984.

[9] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise. A primal-dual interior point algorithm for linear programming, In: N. Megiddo (ed.), *Progress in Mathematical Programming; Interior Point and Related Methods*, pp. 29-47, Springer Verlag, New York, 1989.

[10] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise. A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems, *Lecture Notes in Computer Science*, Volume 538, Springer-Verlag, Berlin, Germany, 1991.

[11] J. Peng, C. Roos and T. Terlaky. A new class of polynomial primal-dual methods for linear and semidefinite optimization. *European Journal of Operations Research*, 143(2), pp. 234-256, 2002.

[12] J. Peng, C. Roos and T. Terlaky. Self-regular functions and new search directions for linear and semidefinite optimization. *Mathematical Programming*, 93, pp. 129–171, 2002.

[13] C. Roos, T. Terlaky and J.-Ph.Vial. *Theory and Algorithms for Linear Optimization. An Interior-Point Approach*, John Wiley & Sons, Chichester, UK, 1997.

[14] S. J. Wright. *Primal-Dual Interior-Point Methods*, SIAM, Philadelphia, USA, 1997.

[15] Y. Ye. *Interior point algorithms, Theory and Analysis*, John Wiley & Sons, Chichester, UK, 1997.