Newton's Method for the Ellipsoidal l_p Norm Facility Location Problems

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Abstract

We use the ellipsoidal l_p norm to measure the distance for both the desirable and the obnoxious facility location problems with constraints. The optimality conditions for both location models are derived. We give Newton's method for the models. Finally, numerical examples are presented.

1 Introduction

Denote the l_p norm (Minkowski distance of order p, $p > 1$) of a d-dimensional Euclidean space $\mathbb{R}^d$ as

$$l_p(\mathbf{z}) \overset{\text{def}}{=} \left[\sum_{i=1}^{d} |z_i|^p \right]^{1/p} .$$

The approximation of road distances by the weighted l_p distance measure has been studied in a series of papers ([5, 6], etc.) and it is argued with empirical study that l_p distances weighted by an inflation factor tailored to given regions can better describe the irregularity in the transportation networks such as hill, bends, and are therefore superior to the weighted rectangular and Euclidean norms.

In this paper, we measure the distances between facilities by the ellipsoidal l_p norm distance:

$$l_{pM}(\mathbf{z}) \overset{\text{def}}{=} l_p(M\mathbf{z}) ,$$

where M is a linear transformation. It is not hard to see that l_{pM} is a norm when M is nonsingular. Note that the l_{pM} distance is the l_p norm when M is the identity. And it includes the Euclidean distance (2-norm distance), the Manhattan distance (l_1 norm distance), and the Chebyshev distance (l_∞ norm distance). It also includes the l_{pb} norm distance [4]:

$$l_{pb}(\mathbf{z}) \overset{\text{def}}{=} \left[\sum_{i=1}^{n} b_i |x_i|^p \right]^{1/p} , \quad b_i > 0 \, (i = 1, \ldots, n) .$$

The l_{pM} distance measure can better describe the actual transportation networks than the weighted l_p distance or the l_{pb} distance can. Love and Morris [6] took an empirical study to compare the goodness-of-fit of different distance functions, including l_{2M} ($p = 2$) and l_p norms. Their conclusion is that the l_{2M} norm estimates the rural distances best. Their analysis shows that p can describe the degree of rectangular bias, and the l_{2M} can describe the the travel direction of ease. Based on their study, we conclude that the l_{pM} distance measure is superior to the l_{2M} and the l_p distances since it includes both the degree of bias and the direction of bias. Unfortunately, no algorithm even for the unconstrained case is given in [6]. The figures below show the effect of p and M (for $M = \left[\begin{smallmatrix} 1 & 1.5 \\ 3 & 1 \end{smallmatrix}\right]$.) For $M = I$, the travel ease is not affected by direction

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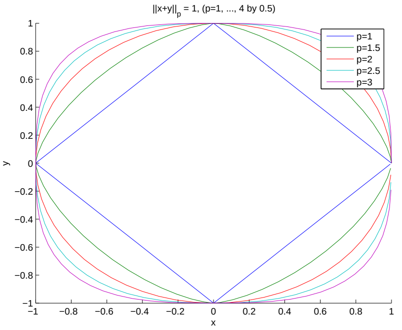
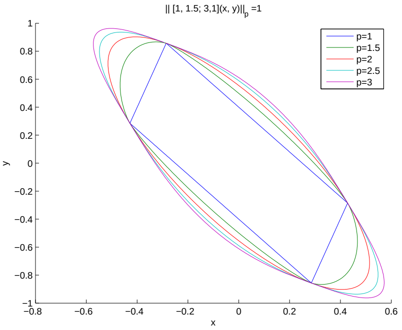
if $p = 2$; when $p = 1$, travel is easiest at 45° to the axes and most difficult parallel to the axes (see [6]) and the ease goes opposite as p becomes larger. With the matrix M, the direction of easiness is not restricted to only axes and the ratio of travel easiness and difficulty can be any number.





It usually takes a long time and a large amount of capital to build a facility, and after its establishment, the facility is supposed to operate for a long period. Therefore, it is important to calculate the location of the new facility as accurate as possible.

In this paper, we consider both desirable and obnoxious facility location problems [2]. For the desirable facility location problem, it is desirable that the sum of the distances between the new facility and existing locations be as short as possible. For the obnoxious location problem, the longer the sum of the distances between the new facility and existing locations the better. See [3] for a survey of the obnoxious location models. Some facilities, such as chemical plants, power plants, air ports, train stations, nuclear reactors, waste dumps, emanate chemical or nuclear pollutants, heat, noise, or magnetic waves. The residents in the region the obnoxious facility to be located desire that this facility to be sit as far away as possible from them. The obnoxiousness of the facility is an inverse factor of the distance to it.

For the desirable location problem, we want to minimize the transportation costs between the new facility and the existing ones. In real word, the road connecting two points is not likely a straight line, but a curve, due to existing landmarks or geographic conditions. The curvature of the road can be better described via the rotation of the axes of the coordinates for the facilities and a proper choice of p. In addition, the transportation costs on different segments of a road may be different due to the physical conditions and traffic flows of different segments. For instance, some segments may consist of bridges or uphills. This can be described by the scaling of corresponding axes of the coordinates. The model is more accurate if we use different p and different transformation matrices for different locations instead of one p and one M for all locations.

The objective of the obnoxious facility location problem is to minimize the effect produced by the pollutants. The longer the pollutants travels, the smaller their negative effects. The dispersion of pollutants are usually affected by meteorology, ground morphology. For instance, hills, tall buildings, rivers, and greens may affect the velocities of winds in different directions in different areas that direct the pollutants to travel in certain paths. As well, forests and mountains may damp the pollution effect.

The new facility is required to be within some region. For the desirable location problem, the distances from the new facility to some resources, such as transportation center, suppliers, should not exceed some upperbounds. For the obnoxious location problem, the distances from the obnoxious facility to some essential locations, such as hospitals, schools, tourism spots, should be above some thresholds r_j, due to some legal or environmental considerations.

The rest of the paper is organized as follows. In §2, we present our model for the ellipsoidal l_p norm facility location problem. In §3, we give the optimality conditions for our model and then reformulate them into a system of equations. In §4, we present our nonsmooth Newton's method for the location problem and give some numerical examples.

2 The Mathematical Models

Let vectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ represent the n existing locations. Let vector $\mathbf{x}$ denote the new facility to be located. In this part, we give both our desirable and obnoxious facility location models.

2.1 The Desirable Model

$$\min_{\mathbf{x}} \quad \sum_{i=1}^{n} w_i \left\| M_i(\mathbf{x} - \mathbf{f}_i) \right\|_{p_i} \tag{1a}$$

$$\text{s.t.} \quad A\mathbf{x} \leq \mathbf{b} \tag{1b}$$

$$\left\| M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j}) \right\|_{p_{n+j}} \leq r_j \quad (j = 1, \ldots, s) \tag{1c}$$

In the above model, $M_i \, (i = 1, \ldots, n+s)$ are d_i-by-d matrices, which are not necessarily nonsingular. The weight $w_i > 0 \, (i = 1, \ldots, n)$ is a combination of demands and inflation factor which counts for the condition of transportation equipments, cost of manpower, etc., for the existing location $\mathbf{f}_i$. Note that w_i doesn't include the factors related to road condition, which are taken care of by the M_i and p_i. $A\mathbf{x} \leq \mathbf{b}$ represents the region in which the new facility to be located.

The constraints (1b) give the region in which the new facility can be located and A is an m-by-d matrix.

The vectors $\mathbf{f}_j \, (j = n, \ldots n+s)$ represent the resources, such as transportation centers. r_i are constants. The constraints (1c) specify that the new facility should be within certain distances from these resources. It is possible that some resources $\mathbf{f}_j \, (n < j \leq n+s)$ are just some existing facilities $\mathbf{f}_i \, (1 \leq i \leq n)$.

Note that (1) is reduced to the ordinary l_p norm facility location problem if all $M_i \, (i = 1, \ldots, n+s)$ are the identity, and the constraints (1b) and (1c) are empty.

2.2 The Obnoxious Location Model

The obnoxious location model we are considering is the following.

$$\max_{\mathbf{x}} \quad \sum_{i=1}^{n} w_i \left\| M_i(\mathbf{x} - \mathbf{f}_i) \right\|_{p_i} \tag{2a}$$

$$\text{s.t.} \quad A\mathbf{x} \leq \mathbf{b} \tag{2b}$$

$$\left\| M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j}) \right\|_{p_{n+j}} \geq r_j \quad (j = 1, \ldots, s) \tag{2c}$$

In the above model, the vectors $\mathbf{f}_j \ (j = 1, \ldots, s)$ denote the essential locations from which the pollution effect should be within some threshold. Some of them may or may not be the same as the existing sites $\mathbf{f}_i \ (i = 1, \ldots, n)$ included in the maxisum objective. w_i is the weight on location $\mathbf{f}_i \ (i = 1, \ldots, n)$, which is associated with the number and importance of residents there and doesn't include the factors on distance measure. For some type of residents, such as patients, children, the weight might be higher than that for others.

3 Optimality Conditions

In this section, we derive the optimality conditions for both (1) and (2), and reformulate them into a system of equations.

3.1 The Desirable Model

We first show that (1) is a convex program in order to give necessary and sufficient conditions for optimality.

For any $0 \leq \lambda \leq 1$, and $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^d$, by Minkowski's inequality, we have

$$\begin{aligned}\sum_{i=1}^{n} w_i \left\| M_i \left[\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 - \mathbf{f}_i \right] \right\|_{p_i} &= \sum_{i=1}^{n} w_i \left\| \lambda M_i(\mathbf{x}_1 - \mathbf{f}_i) + (1-\lambda) M_i(\mathbf{x}_2 - \mathbf{f}_i) \right\|_{p_i} \\ &\leq \lambda \sum_{i=1}^{n} w_i \left\| M_i(\mathbf{x}_1 - \mathbf{f}_i) \right\|_{p_i} + (1-\lambda) \sum_{i=1}^{n} w_i \left\| M_i(\mathbf{x}_2 - \mathbf{f}_i) \right\|_{p_i}.\end{aligned}$$

Therefore, the objective function (1a) is convex in $\mathbf{x}$.

For $i = 1, \ldots, n+s$, define q_i as the scalar satisfying $\frac{1}{q_i} + \frac{1}{p_i} = 1$.

Nest, we consider the Lagrangian dual to (1).

We use z_{il} to denote the lth element of the vector $\mathbf{z}_i$, and M_{il} to denote the lth row of the matrix M_i. Define the vectors $\boldsymbol{\eta} \overset{\text{def}}{=} (\eta_1, \dots, \eta_s)^T$, $\boldsymbol{\lambda} \overset{\text{def}}{=} (\lambda_1, \dots, \lambda_m)^T$.

By Hölder's inequality, when $M_i(\mathbf{x} - \mathbf{f}_i) \neq \mathbf{0}$,

$$w_i \left\| M_i(\mathbf{x} - \mathbf{f}_i) \right\|_{p_i} \geq_{(\|\mathbf{z}_i\|_{q_i} \leq w_i)} \mathbf{z}_i^T M_i(\mathbf{x} - \mathbf{f}_i)$$

with equality holds iff

$$\|\mathbf{z}_i\|_{q_i} = w_i, \quad \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i}^{p_i} |z_{il}|^{q_i} = w_i^{q_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i}, \quad \operatorname{sign}(\mathbf{z}_{il}) = \operatorname{sign}\left[M_{il}(\mathbf{x} - \mathbf{f}_i)\right] .$$

When $M_i(\mathbf{x} - \mathbf{f}_i) = \mathbf{0}$, the subdifferential of $\|M_i(\mathbf{x} - \mathbf{f}_i)\|$ is $\{M_i^T \mathbf{z}_i \colon \|\mathbf{z}_i\|_{q_i} \leq 1\}$.

Hence, we have that the dual to (1) is

$$\begin{aligned}
&\max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \sum_{i=1}^{n} w_i \left\| M_i(\mathbf{x} - \mathbf{f}_i) \right\|_{p_i} + \boldsymbol{\lambda}^T (A\mathbf{x} - \mathbf{b}) + \sum_{j=1}^{s} \eta_j \Bigg(\left\| M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j}) \right\|_{p_{n+j}} \\
&\qquad - r_j \Bigg) = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \max_{\substack{\|\mathbf{z}_i\|_{q_i} \leq w_i \\ (i=1,\dots,n) \\ \|\mathbf{z}_j\|_{q_j} \leq \eta_j \\ (j=n+1,\dots,n+s)}} \Bigg[\sum_{i=1}^{n} \mathbf{z}_i^T M_i(\mathbf{x} - \mathbf{f}_i) + \boldsymbol{\lambda}^T (A\mathbf{x} - \mathbf{b}) \\
&\qquad + \sum_{j=1}^{s} \mathbf{z}_{j+n}^T M_{j+n}(\mathbf{x} - \mathbf{f}_{j+n}) - \sum_{j=1}^{s} \eta_j r_j \Bigg] = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \max_{\substack{\|\mathbf{z}_i\|_{q_i} \leq w_i \ (i=1,\dots,n) \\ \|\mathbf{z}_j\|_{q_j} \leq \eta_j \ (j=n+1,\dots,n+s)}} \\
&\qquad \left[\left(\sum_{i=1}^{n+s} \mathbf{z}_i^T M_i + \boldsymbol{\lambda}^T A \right) \mathbf{x} - \sum_{i=1}^{n+s} \mathbf{z}_i^T M_i \mathbf{f}_i - \boldsymbol{\lambda}^T \mathbf{b} - \sum_{j=1}^{s} \eta_j r_j \right].
\end{aligned}$$

Let $\mathbf{z}(\mathbf{x})$ be the optimal solution of $\mathbf{z}$ to the inner maximization problem for a given $\mathbf{x}$. The first term in the above expression implies $\sum_{i=1}^{n+s} \mathbf{z}(\mathbf{x})_i^T M_i + \boldsymbol{\lambda}^T A = \mathbf{0}$; otherwise the value of the second minimization is unbounded.

Therefore, the Lagrangian dual to (1) is:

$$\begin{array}{ll}
\min_{\boldsymbol{\lambda}, \boldsymbol{\eta}, \mathbf{z}_i} & \sum_{i=1}^{n+s} \mathbf{z}_i^T M_i \mathbf{f}_i + \boldsymbol{\lambda}^T \mathbf{b} + \sum_{j=1}^{s} \eta_j r_j \\
\text{s.t.} & \sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \\
& \|\mathbf{z}_i\|_{q_i} \leq w_i \quad (i = 1, \dots, n) \\
& \|\mathbf{z}_j\|_{q_j} \leq \eta_j \quad (j = n+1, \dots, n+s) \\
& \boldsymbol{\lambda} \geq \mathbf{0} \\
& \boldsymbol{\eta} \geq \mathbf{0}
\end{array} \tag{3}$$

The dual of the linearly constrained l_p norm facility location problem has been studied in [7]; however, our model includes the resource constraints (1c).

Assume there is a strict interior solution, i.e., $\exists\, \tilde{\mathbf{x}} \in \mathbb{R}^d$ such that $A\tilde{\mathbf{x}} \leq \mathbf{b}$ and $\|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j} < r_j$ ($j = n+1, \dots, n+s$). Then there is no duality gap between (3) and (1) (see for instance [9]), since the objective and the constraints are convex.

In addition, at optimality, $\mathbf{x}, \mathbf{z}_i (i = 1, \ldots, n+s), \boldsymbol{\lambda}, \boldsymbol{\eta}$ satisfy the following conditions:

$$
\begin{gathered}
\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \ , \\
\eta_{j-n} \left(r_{j-n}^{p_j} - \|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j}^{p_j} \right) = 0 \quad (j = n+1, \ldots, n+s) \ , \\
\eta_j \geq 0 \quad (j = 1, \ldots, s) \ , \\
\|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \leq r_j \quad (j = 1, \ldots, s) \ , \\
\lambda_i (b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \ldots, m) \ , \\
\lambda_i \geq 0 \quad (i = 1, \ldots, m) \ , \\
A_i \mathbf{x} \leq b_i \quad (i = 1, \ldots, m) \ , \\
\alpha_i |z_{il}|^{q_i} = |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \quad (i = 1, \ldots, n; \ l = 1, \ldots, d_i) \ , \\
\alpha_j |z_{jl}|^{q_j} = |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \quad (j = 1+n, \ldots, n+s; \ l = 1, \ldots, d_j) \ , \\
\operatorname{sign}(z_{il}) = \operatorname{sign}\left[M_{il}(\mathbf{x} - \mathbf{f}_i)\right] \quad (i = 1, \ldots, n+s; \ l = 1, \ldots, d_i) \ , \\
\alpha_i (w_i - \|\mathbf{z}_{il}\|_{p_i}) = 0 \quad (i = 1, \ldots, n) \ , \\
\|\mathbf{z}_i\|_{q_i} \leq w_i \quad (i = 1, \ldots, n) \ , \\
\alpha_{n+j} (\eta_j - \|\mathbf{z}_{n+j}\|_{p_{n+j}}) = 0 \quad (j = 1, \ldots, s) \ , \\
\|\mathbf{z}_{n+j}\|_{q_{n+j}} \leq \eta_j \quad (j = 1, \ldots, s) \ .
\end{gathered}
$$

In the above system, we allow $\operatorname{sign}(0) = \operatorname{sign}(a)$ for all $a \in \mathbb{R}$. We use A_i to denote the ith row of A.

Usually the new facility is not located in the existing demand points or resources. Therefore, we assume the linear constraints $A\mathbf{x} \leq \mathbf{b}$ excluding $\mathbf{f}_i$ $(i = 1, \ldots, n+s)$. Then we have

$$
\|z_i\|_{q_i} = w_i \ (i = 1, \ldots, n) \ , \qquad \|z_j\|_{q_j} = \eta_j \ (j = n+1, \ldots, n+s) \ .
$$

In addition, for each $i = 1, \ldots, n+s$, adding the d_i equations

$$
\alpha_i |\mathbf{z}_{il}|^{q_i} = |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \text{ (for } l = 1, \ldots, d_i),
$$

we obtain

$$
\alpha_i w_i^{q_i} = |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \ (i = 1, \ldots, n), \quad \alpha_i \eta_i^{q_i} = |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \ (i = n+1, \ldots, n+s),
$$

Thus, we can omit the variables α_i $(i = 1, \ldots, n+s)$. And the optimality conditions are reduced to the following.

$$
\begin{gathered}
\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \ , \\
\eta_j \left(r_j^{p_{n+j}} - \|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}}^{p_{n+j}} \right) = 0 \quad (j = 1, \ldots, s) \ , \\
\eta_j \geq 0 \quad (j = 1, \ldots, s) \ , \\
\|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \leq r_j \quad (j = 1, \ldots, s) \ , \\
\lambda_i (b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \ldots, m) \ , \\
\lambda_i \geq 0 \quad (i = 1, \ldots, m) \ , \\
A\mathbf{x} \leq \mathbf{b} \ , \\
\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \right)^{\frac{1}{q_i}} |z_{il}| = w_i \, |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{\frac{p_i}{q_i}} \quad (i = 1, \ldots, n; \ l = 1, \ldots, d_i) \ , \\
\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \right)^{\frac{1}{q_j}} |z_{jl}| = \eta_{j-n} \, |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{\frac{p_j}{q_j}} \quad (j = n+1, \ldots, n+s; \ l = 1, \ldots, d_j) \ , \\
\operatorname{sign}(z_{il}) = \operatorname{sign}\left[M_{il}(\mathbf{x} - \mathbf{f}_i)\right] \quad (i = 1, \ldots, n+s; \ l = 1, \ldots, d_i) \ .
\end{gathered}
$$

However, the above system is not easy to solve, because it includes both equalities and inequalities. Next, we use the nonlinear complementarity function min to reformulate the complementarity conditions in order to transform the above system into a system of equations, which can be solved by Newton's method. Note that $min(a, b) = 0$ is equivalent to $a \geq 0$, $b \geq 0$, and at least one of a, b is 0.

Also note that when $p \leq q$, the function $|x|^{p/q}$ is nondifferentiable at $x = 0$. The second and third last equations in the above system involve such functions. To avoid these nonsmooth points, we distinguish between $p_i \geq 2$ and $p_i < 2$ in our formula.

Notice that $p_i \geq 2 \Rightarrow q_i \leq 2$, $p_i \leq 2 \Rightarrow q_i \geq 2$.

From $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $(i = 1, \ldots, n + s)$, we have

$$\frac{p_i}{q_i} = p_i - 1 = \frac{1}{q_i - 1}, \quad \frac{q_i}{p_i} = q_i - 1 = \frac{1}{p_i - 1} \ .$$

Below is the system of equations we obtain.

$$\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \tag{4a}$$

$$\min \left[\eta_{j-n}, r_{j-n}^{p_j} - \|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j}^{p_j}\right] = 0 \quad (j = n + 1, \ldots, n + s) \ , \tag{4b}$$

$$\min (\lambda_i, b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \ldots, m) \ , \tag{4c}$$

$$\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i}\right)^{\frac{1}{q_i}} z_{il} - w_i \, |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{\frac{p_i}{q_i}} \operatorname{sign}(M_{il}(\mathbf{x} - \mathbf{f}_i)) = 0 \qquad (i \in \{1, \ldots, n\}, p_i \geq 2;\ l = 1, \ldots, d_i) \tag{4d}$$

$$\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i}\right)^{\frac{1}{p_i}} |z_{il}|^{\frac{q_i}{p_i}} \operatorname{sign}(z_{il}) - w_i^{\frac{q_i}{p_i}} M_{il}(\mathbf{x} - \mathbf{f}_i) = 0 \qquad (i \in \{1, \ldots, n\}, p_i < 2;\ l = 1, \ldots, d_i) \tag{4e}$$

$$\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j}\right)^{\frac{1}{q_j}} z_{jl} - \eta_{j-n} \, |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{\frac{p_j}{q_j}} \operatorname{sign}(M_{jl}(\mathbf{x} - \mathbf{f}_j)) = 0 \qquad (j \in \{n + 1, \ldots, n + s\}, p_j \geq 2;\ l = 1, \ldots, d_j) \tag{4f}$$

$$\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j}\right)^{\frac{1}{p_j}} |z_{jl}|^{\frac{q_j}{p_j}} \operatorname{sign}(z_{jl}) - \eta_{j-n}^{\frac{q_j}{p_j}} M_{jl}(\mathbf{x} - \mathbf{f}_j) = 0 \qquad (j \in \{n + 1, \ldots, n + s\}, p_j < 2;\ l = 1, \ldots, d_j) \tag{4g}$$

3.2 The Obnoxious Location Model

In this part, we give optimality conditions for the obnoxious location model and then reformulate them into nonlinear equations.

By the results in the previous part, the objective of the obnoxious location is to maximize a convex location in a nonconvex region. Therefore, this problem is harder than the desirable one.

We first give the dual to (2). By Hölder's inequality,

$$\max_{\mathbf{x}} w_i \, \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i} = \min_{(\|\mathbf{z}_i\|_{q_i} \geq w_i)} \max_{\mathbf{x}} \mathbf{z}_i^T M_i(\mathbf{x} - \mathbf{f}_i) \ ,$$

where $\|\mathbf{z}_i\|_{q_i} = w_i$, $\|M_i(\mathbf{x}-\mathbf{f}_i)\|_{p_i}^{p_i} |z_{il}|^{q_i} = w_i^{q_i} |M_{il}(\mathbf{x}-\mathbf{f}_i)|^{p_i}$, $\operatorname{sign}(\mathbf{z}_{il}) = \operatorname{sign}[M_{il}(\mathbf{x} - \mathbf{f}_i)]$ when $M_i(\mathbf{x}-\mathbf{f}_i) \neq \mathbf{0}$; and $\|\mathbf{z}_i\|_{q_i} \geq w_i$ when $M_i(\mathbf{x} - \mathbf{f}_i) = \mathbf{0}$.

Hence, we have that the dual to (2) is the following.

$$\begin{aligned}
&\min_{\boldsymbol{\lambda}\geq\mathbf{0},\boldsymbol{\eta}\geq\mathbf{0}}\max_{\mathbf{x}}\sum_{i=1}^{n} w_i \left\|M_i(\mathbf{x}-\mathbf{f}_i)\right\|_{p_i} - \boldsymbol{\lambda}^T(A\mathbf{x}-\mathbf{b}) + \sum_{j=1}^{s}\eta_j\Big(\left\|M_{n+j}(\mathbf{x}-\mathbf{f}_{n+j})\right\|_{p_{n+j}} \\
&\qquad - r_j\Big) = \min_{\boldsymbol{\lambda}\geq\mathbf{0},\boldsymbol{\eta}\geq\mathbf{0}}\ \min_{\substack{\|\mathbf{z}_i\|_{q_i}\geq w_i\\(i=1,\dots,n)\\ \|\mathbf{z}_j\|_{q_j}\geq\eta_j\\(j=n+1,\dots,n+s)}}\ \max_{\mathbf{x}}\left[\sum_{i=1}^{n}\mathbf{z}_i^T M_i(\mathbf{x}-\mathbf{f}_i) - \boldsymbol{\lambda}^T(A\mathbf{x}-\mathbf{b})\right. \\
&\qquad \left. + \sum_{j=1}^{s}\mathbf{z}_{j+n}^T M_{j+n}(\mathbf{x}-\mathbf{f}_{j+n}) - \sum_{j=1}^{s}\eta_j r_j\right] = \min_{\boldsymbol{\lambda}\geq\mathbf{0},\boldsymbol{\eta}\geq\mathbf{0}}\ \min_{\substack{\|\mathbf{z}_i\|_{q_i}\geq w_i\ (i=1,\dots,n)\\ \|\mathbf{z}_j\|_{q_j}\geq\eta_j\ (j=n+1,\dots,n+s)}}\ \max_{\mathbf{x}} \\
&\qquad\qquad \left[\left(\sum_{i=1}^{n+s}\mathbf{z}_i^T M_i - \boldsymbol{\lambda}^T A\right)\mathbf{x} - \sum_{i=1}^{n+s}\mathbf{z}_i^T M_i\mathbf{f}_i + \boldsymbol{\lambda}^T\mathbf{b} - \sum_{j=1}^{s}\eta_j r_j\right].
\end{aligned}$$

In the above expression, $\sum_{i=1}^{n+s}\mathbf{z}_i^T M_i = \boldsymbol{\lambda}^T A$; otherwise, the inner maximization would be unbounded. Therefore, the dual to (2) is

$$\begin{array}{ll}
\min_{\boldsymbol{\lambda},\boldsymbol{\eta},\mathbf{z}_i} & -\sum_{i=1}^{n+s}\mathbf{z}_i^T M_i\mathbf{f}_i + \boldsymbol{\lambda}^T\mathbf{b} - \sum_{j=1}^{s}\eta_j r_j \\
\text{s.t.} & \sum_{i=1}^{n+s} M_i^T\mathbf{z}_i - A^T\boldsymbol{\lambda} = \mathbf{0} \\
 & \|\mathbf{z}_i\|_{q_i}\geq w_i \quad (i=1,\dots,n) \\
 & \|\mathbf{z}_j\|_{q_j}\geq \eta_j \quad (j=n+1,\dots,n+s) \\
 & \boldsymbol{\lambda}\geq\mathbf{0} \\
 & \boldsymbol{\eta}\geq\mathbf{0}
\end{array} \tag{5}$$

In addition, we have $\|\mathbf{z}_i\|_{q_i}=w_i$, $\|M_i(\mathbf{x}-\mathbf{f}_i)\|_{p_i}^{p_i}|z_{il}|^{q_i}=w_i^{q_i}|M_{il}(\mathbf{x}-\mathbf{f}_i)|^{p_i}$, $\operatorname{sign}(\mathbf{z}_{il})=\operatorname{sign}\left[M_{il}(\mathbf{x}-\mathbf{f}_i)\right]$ when $M_i(\mathbf{x}-\mathbf{f}_i)\neq\mathbf{0}$ for $i=1,\dots,n$. And $\|\mathbf{z}_j\|_{q_j}=\eta_j$, $\|M_j(\mathbf{x}-\mathbf{f}_j)\|_{p_j}^{p_j}|z_{jl}|^{q_j}=\eta_j^{q_j}|M_{jl}(\mathbf{x}-\mathbf{f}_j)|^{p_j}$, $\operatorname{sign}(\mathbf{z}_{jl})=\operatorname{sign}\left[M_{jl}(\mathbf{x}-\mathbf{f}_j)\right]$ when $M_i(\mathbf{x}-\mathbf{f}_j)\neq\mathbf{0}$ for $j=1,\dots,s$. We also have $\lambda_i(A_i\mathbf{x}-b_i)=0\,(i=1,\dots,m)$, $\eta_j\left(\|M_{n+j}(\mathbf{x}-\mathbf{f}_{n+j})\|_{p_{n+j}}-r_j\right)=0$, $(j=1,\dots,s)$.

The new facility is required not to be located in existing residence sites. So the objective is differentiable at optimum. Therefore, the reverse convex constraint qualification [1] is satisfied. (see [8]).

We conclude(see [8]). that the necessary local optimality conditions for (2) are:

---

[1] Let X^0 be an open set in $\mathbb{R}^n$, let $\mathbf{g}$ be an m-dimensional vector function defined on X^0, and let $X\overset{\text{def}}{=}\{\mathbf{x}\colon \mathbf{x}\in X^0, \mathbf{g}(\mathbf{x})\leq\mathbf{0}\}$. $\mathbf{g}$ is said to satisfy the reverse convex constraint qualification at $\bar{\mathbf{x}}\in X$ if $\mathbf{g}$ is differentiable and for each $i\in\{i\colon g_i(\bar{\mathbf{x}})=0\}$, either g_i is concave at $\bar{\mathbf{x}}$ or g_i is linear on $\mathbb{R}^n$.

$$
\begin{gathered}
\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i - A^T \boldsymbol{\lambda} = \mathbf{0} \ , \\
\eta_{j-n}\left(r_{j-n}^{p_j} - \|M_j(\mathbf{x}-\mathbf{f}_j)\|_{p_j}^{p_j}\right) = 0 \quad (j = n+1,\ldots,n+s) \ , \\
\eta_j \geq 0 \quad (j = 1,\ldots,s) \ , \\
\|M_{n+j}(\mathbf{x}-\mathbf{f}_{n+j})\|_{p_{n+j}} \geq r_j \quad (j = 1,\ldots,s) \ , \\
\lambda_i(b_i - A_i\mathbf{x}) = 0 \quad (i = 1,\ldots,m) \ , \\
\lambda_i \geq 0 \quad (i = 1,\ldots,m) \ , \\
A_i\mathbf{x} \leq b_i \quad (i = 1,\ldots,m) \ , \\
\alpha_i |z_{il}|^{q_i} = |M_{il}(\mathbf{x}-\mathbf{f}_i)|^{p_i} \quad (i = 1,\ldots,n;\ l = 1,\ldots,d_i) \ , \\
\alpha_j |z_{jl}|^{q_j} = \eta_{j-n}\, |M_{jl}(\mathbf{x}-\mathbf{f}_j)|^{p_j} \quad (j = 1+n,\ldots,n+s;\ l = 1,\ldots,d_j) \ , \\
\operatorname{sign}(z_{il}) = \operatorname{sign}\left[M_{il}(\mathbf{x}-\mathbf{f}_i)\right] \quad (i = 1,\ldots,n+s;\ l = 1,\ldots,d_i) \ , \\
\alpha_i(w_i - \|\mathbf{z}_{il}\|_{p_i}) = 0 \quad (i = 1,\ldots,n) \ , \\
\|\mathbf{z}_i\|_{q_i} \geq w_i \quad (i = 1,\ldots,n) \ , \\
\alpha_{n+j}(\eta_j - \|\mathbf{z}_{n+j}\|_{p_{n+j}}) = 0 \quad (j = 1,\ldots,s) \ , \\
\|\mathbf{z}_{n+j}\|_{q_{n+j}} \geq \eta_j \quad (j = 1,\ldots,s) \ .
\end{gathered}
$$

As before, we assume $\operatorname{sign}(0) = \operatorname{sign}(a)$, for all $a \in \mathbb{R}$. We also use the min to reformulate the complementarity.

We assume that $\mathbf{f}_i$ $(i = 1,\ldots,n+s)$ doesn't satisfy $A\mathbf{x} \leq \mathbf{b}$. This means that the region the obnoxious facility to be sit doesn't include existing residence sites.

In the formula, we also distinguish between $p_i \geq 2$ and $p_i < 2$ to avoid some nondifferentiable points. We then transform the optimality conditions into the following system of equations.

$$
\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i - A^T \boldsymbol{\lambda} = \mathbf{0} \tag{6a}
$$

$$
\min\left[\eta_{j-n}, \|M_j(\mathbf{x}-\mathbf{f}_j)\|_{p_j}^{p_j} - r_{j-n}^{p_j}\right] = 0 \quad (j = n+1,\ldots,n+s) \ , \tag{6b}
$$

$$
\min\left(\lambda_i, b_i - A_i\mathbf{x}\right) = 0 \quad (i = 1,\ldots,m) \ , \tag{6c}
$$

$$
\begin{gathered}
\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x}-\mathbf{f}_i)|^{p_i}\right)^{\frac{1}{q_i}} z_{il} - w_i\, |M_{il}(\mathbf{x}-\mathbf{f}_i)|^{\frac{p_i}{q_i}} \operatorname{sign}\left(M_{il}(\mathbf{x}-\mathbf{f}_i)\right) = 0 \\
(i \in \{1,\ldots,n\}, p_i \geq 2;\ l = 1,\ldots,d_i)
\end{gathered} \tag{6d}
$$

$$
\begin{gathered}
\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x}-\mathbf{f}_i)|^{p_i}\right)^{\frac{1}{p_i}} |z_{il}|^{\frac{q_i}{p_i}} \operatorname{sign}(z_{il}) - w_i^{\frac{q_i}{p_i}} M_{il}(\mathbf{x}-\mathbf{f}_i) = 0 \\
(i \in \{1,\ldots,n\}, p_i < 2;\ l = 1,\ldots,d_i)
\end{gathered} \tag{6e}
$$

$$
\begin{gathered}
\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x}-\mathbf{f}_j)|^{p_j}\right)^{\frac{1}{q_j}} z_{jl} - \eta_{j-n}\, |M_{jl}(\mathbf{x}-\mathbf{f}_j)|^{\frac{p_j}{q_j}} \operatorname{sign}\left(M_{jl}(\mathbf{x}-\mathbf{f}_j)\right) = 0 \\
(j \in \{n+1,\ldots,n+s\}, p_j \geq 2;\ l = 1,\ldots,d_j)
\end{gathered} \tag{6f}
$$

$$
\begin{gathered}
\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x}-\mathbf{f}_j)|^{p_j}\right)^{\frac{1}{p_j}} |z_{jl}|^{\frac{q_j}{p_j}} \operatorname{sign}(z_{jl}) - \eta_{j-n}^{\frac{q_j}{p_j}} M_{jl}(\mathbf{x}-\mathbf{f}_j) = 0 \\
(j \in \{n+1,\ldots,n+s\}, p_j < 2;\ l = 1,\ldots,d_j)
\end{gathered} \tag{6g}
$$

4 Numerical Experiments

In this part, we first describe the algorithm, then give some numerical examples.

4.1 Algorithm

We have implemented the Newton's method for both of the above facility location problems in MATLAB. Newton's method converges fast near an optimal solution and is good for warm starting perturbed instances.

Let F represent the left-hand-side of (4) or (6). Let $\Psi \overset{\text{def}}{=} \frac{F'F}{2}$. Any global optimization method that locates a global minimal solution to Ψ solves (1) or gives a local solution to (2).

We extend the gradient decent method with perturbed nonmonotone line search to skip the nonsmooth points to find a global minimum of Ψ [10].

We adopt the suggested parameters in [1]. The machine accuracy is $\epsilon = 2.2204e - 16$. Our computer program stops either $\|F\|_\infty < opt = \epsilon^{1/3} = 6.0555e-5$, or the infinity norm of the Newton's direction is less than $steptol = \epsilon^{2/3}$; or the number of iterations exceeds $itlimit = 100$.

Denote $\mathbf{w} \overset{\text{def}}{=} (\mathbf{x}; \boldsymbol{\lambda}; \boldsymbol{\eta}; \mathbf{z})$. Let $\Delta\,\mathbf{w}$ represent the Newton's direction to (4) or (6).

The Algorithm

```
**Initialization** Set constants $s > 0$, $0 < \sigma < 1$, $\beta \in (0,1), \gamma \in (\beta, 1)$, $nml \geq 1$. For each $k \geq 0$, assume $\Psi$ is
    differentiable at $\mathbf{w}^k$. Set $k = 0$.

**Do while** $\|F\|_\infty \geq opt$, $\left\|\mathbf{w}^{k+1} - \mathbf{w}^k\right\|_\infty \geq steptol$, and $k \leq itlimit$.

    1. Find the Newton's direction for (4) or (6): $\Delta\,\mathbf{w}^k$.
    2. (a) Set $\alpha^{k,0} = s$, i = 0.
       (b) Find the smallest nonnegative integer $l$ for which
               $\Psi(\mathbf{w}^k) - \Psi\left(\mathbf{w}^k + \beta^l \alpha^{k,i}\,\Delta\,\mathbf{w}^k\right) \geq_{0 \leq j \leq m(k)} -\sigma \beta^l \alpha^{k,i}\,\nabla\,\Psi(\mathbf{w}^j)^T\,\Delta\,\mathbf{w}^k.$
           where $m(0) = 0$ and $0 \leq m(k) \leq \min[m(k-1)+1, nml]$.
       (c) If $\Psi$ is nondifferentiable at $\left(\mathbf{w}^k + \beta^l \alpha^{k,i}\,\Delta\,\mathbf{w}^k\right)$, find $t \in [\gamma, 1)$ so that $\Psi$ is differentiable at
           $\left(\mathbf{w}^k + t\beta^l \alpha^{k,i}\,\Delta\,\mathbf{w}^k\right)$, set $\alpha^{k,i+1} = t\beta^l\alpha^{k,i}$, $i+1 \to i$, go to step 2b.
           Otherwise, set $\alpha^k = \beta^l \alpha^{k,i}$, $\mathbf{w}^{k+1} = \mathbf{w}^k + \alpha^k\,\Delta\,\mathbf{w}$, $k+1 \to k$.
```

4.2 An Example

We adopt the suggested parameters in [1]. The machine accuracy of the computer running the code is $\epsilon = 2.2204e-16$. Our computer program stops either $\|F\|_\infty < opt = \epsilon^{1/3} = 6.0555e-5$, or the infinity norm of the Newton's direction is less than $steptol = \epsilon^{2/3}$; or the number of iterations exceeds $itlimit = 100$. We set $s = 1$, $\beta = 0.5$, $\sigma = 1.0e-4$, $nml = 10$. Below are numerical examples for both the desirable model and the obnoxious model with 10 existing facilities and 2 resources:

4.2.1 The Desirable Model

$f_1 = (0.8175669, 0.9090309, 0.2491902)$, $f_2 = (0.3332802, 0.4928804, 0.0171481)$, $f_3 = (0.5420494, 0.8212501, 0.3767346)$, $f_4 = (0.5911344, 0.9217661, 0.6447813)$, $f_5 = (0.2692597, 0.2433543, 0.6320366)$, $f_6 = (0.9949809, 0.8636484, 0.3300618)$, $f_7 = (0.5093652, 0.9573201, 0.1044826)$, $f_8 = (0.2043801, 0.8580418, 0.2739731)$, $f_9 = (0.1839796, 0.2708408, 0.7940208)$, $f_{10} = (0.4267598, 0.0453742, 0.1054062)$, $f_{11} = (0.2745187, 0.5993611, 0.1221155)$, $f_{12} = (0.2181376, 0.9671702, 0.5442553)$

The weights for $\mathbf{f}_1, \ldots, \mathbf{f}_{10}$ are: $\mathbf{w} = (0.7781196, 0.2842939, 0.0304183, 0.4431445, 0.2179517, 0.5138524, 0.3852747, 0.48$

The p for $\mathbf{f}_1, \ldots, \mathbf{f}_{12}$ are $\mathbf{p} = (1.7, 1.5, 2.5, 2.1, 2.5, 2.1, 2.3, 2.5, 2.5, 2.1, 2.9, 1.9)$.

The transformation matrices for the distance measure between the new facility and the existing facilities and the resources are:

$$M_1 = \begin{bmatrix} 0.4043783 & 0.2878919 & 0.5358692 \\ 0.4152851 & 0.5412418 & 0.2719560 \\ 0.3998512 & 0.3457063 & 0.3102344 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.6493642 & 0.4302950 & 0.8370728 \\ 0.9980712 & 0.2561977 & 0.7632986 \\ 0.2469849 & 0.1924696 & 0.8870779 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 0.3580290 & 0.1878973 & 0.5333254 \\ 0.5734145 & 0.0221892 & 0.1189078 \\ 0.8408229 & 0.0575776 & 0.8342093 \end{bmatrix}$$
$$M_4 = \begin{bmatrix} 0.3855365 & 0.5720651 & 0.8868140 \\ 0.0013690 & 0.1720985 & 0.6703435 \\ 0.5097000 & 0.4759156 & 0.7431721 \end{bmatrix}$$
$$M_5 = \begin{bmatrix} 0.1805674 & 0.2649339 & 0.6593017 \\ 0.9350963 & 0.9854972 & 0.6605832 \\ 0.9342599 & 0.8032064 & 0.4336484 \end{bmatrix}$$
$$M_6 = \begin{bmatrix} 0.6954798 & 0.2586548 & 0.0355097 \\ 0.7623554 & 0.3843390 & 0.4729695 \\ 0.4053601 & 0.0080771 & 0.2474898 \end{bmatrix}$$
$$M_7 = \begin{bmatrix} 0.0132654 & 0.9779720 & 0.0447804 \\ 0.6306576 & 0.7188327 & 0.8326453 \\ 0.5479869 & 0.5564485 & 0.2328634 \end{bmatrix}$$
$$M_8 = \begin{bmatrix} 0.1321284 & 0.6354911 & 0.6550676 \\ 0.4827887 & 0.5106806 & 0.1679761 \\ 0.5706509 & 0.5329994 & 0.1684392 \end{bmatrix}$$
$$M_9 = \begin{bmatrix} 0.5942561 & 0.9369910 & 0.3738004 \\ 0.8647487 & 0.7414568 & 0.2842745 \\ 0.1066821 & 0.7481810 & 0.2772064 \end{bmatrix}$$
$$M_{10} = \begin{bmatrix} 0.7579091 & 0.6877772 & 0.4022239 \\ 0.1689570 & 0.2506596 & 0.7048818 \\ 0.1025312 & 0.9542088 & 0.6352403 \end{bmatrix}$$
$$M_{11} = \begin{bmatrix} 0.0136439 & 0.2677678 & 0.4533213 \\ 0.8115766 & 0.2669812 & 0.5141040 \\ 0.2874519 & 0.7550024 & 0.1576553 \end{bmatrix}$$
$$M_{12} = \begin{bmatrix} 0.8747555 & 0.7809548 & 0.8933810 \\ 0.8366846 & 0.5110550 & 0.1887794 \\ 0.8529809 & 0.5310465 & 0.3032273 \end{bmatrix}.$$

The data for the feasible region are

$$A = \begin{pmatrix} -0.3725763 & -0.6852143 & -0.2041526 \\ -0.9844071 & -0.2049625 & -0.8364794 \\ 0.4568025 & 0.3087572 & 0.0137238 \end{pmatrix},$$

$\mathbf{b} = (-0.8759974, -1.2869180, 0.5790744)^T$.

The upper bounds for the distances are $\mathbf{r} = (0.6361848, 0.5080279)^T$.

We start from $\mathbf{x} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$, $\boldsymbol{\eta} = \mathbf{0}$. Iterations are depicted in the figure "Example 1".

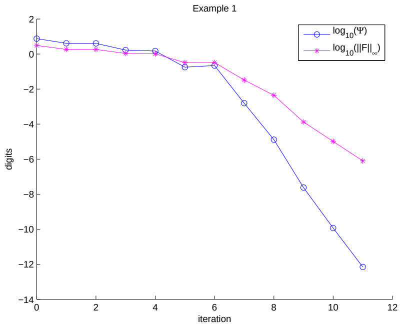
We then perturb each element of $\mathbf{b}$, A, $\mathbf{f}$, randomly by a real number in $(-0.5, 0.5)$ respectively to

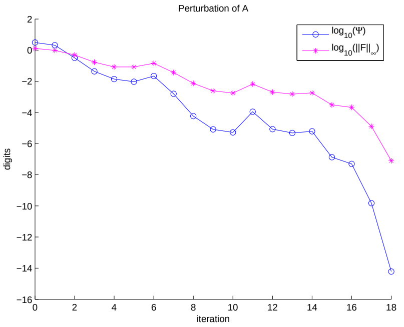
$$b = (-0.4346635, -0.8455841, 1.0204083)^T \ ,$$

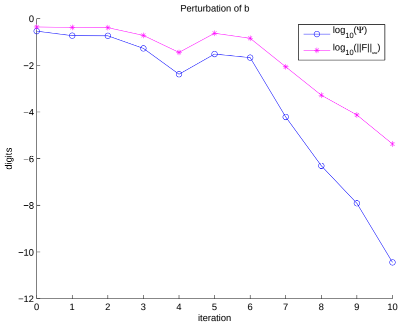
$$A = \begin{pmatrix} -0.6963649 & -1.0090028 & -0.5279412 \\ -1.3081957 & -0.5287511 & -1.1602679 \\ 0.1330139 & -0.0150314 & -0.3100648 \end{pmatrix} \ ,$$

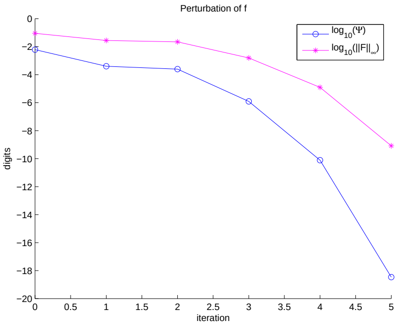
$f(1) = (0.8825418, 0.9740059, 0.3141651)$ $f(2) = (0.3982551, 0.5578554, 0.0821231)$ $f(3) = (0.6070244, 0.8862251, 0.441$
$f(4) = (0.6561094, 0.9867411, 0.7097563)$ $f(5) = (0.3342346, 0.3083293, 0.6970116)$ $f(6) = (1.0599559, 0.9286234, 0.39503$
$f(7) = (0.5743401, 1.0222950, 0.1694576)$ $f(8) = (0.2693551, 0.9230168, 0.3389480)$ $f(9) = (0.2489546, 0.3358158, 0.85899$
$f(10) = (0.4917348, 0.1103492, 0.1703812)$ $f(11) = (0.3394937, 0.6643360, 0.1870905)$ $f(12) = (0.2831126, 1.0321451, 0.60$

The starting point for each perturbed problem is the solution to "Example 1". The iterations are described in the figures: "Perturbation of A", "Perturbation of b", and "Perturbation of f".


Example 1


Perturbation of A


Perturbation of b


Perturbation of f

4.2.2 The Obnoxious Model

The 10 existing facilities are:
$f_1 = (0.8351697, 0.9708701, 0.4337257)$, $f_2 = (0.5029927, 0.4754272, 0.7399495)$, $f_3 = (0.9887191, 0.4000630, 0.6804281)$, $f_4 = (0.9093165, 0.5075840, 0.8947370)$, $f_5 = (0.4425517, 0.3319756, 0.0674839)$, $f_6 = (0.5963061, 0.1664579, 0.1948914)$, $f_7 = (0.7186201, 0.7451580, 0.5479852)$, $f_8 = (0.4763083, 0.9978569, 0.5943900)$, $f_9 = (0.4766319, 0.0927731, 0.4870974)$, $f_{10} = (0.1512887, 0.2611954, 0.6834821)$.

The 2 essential facilities are
$f_{11} = (0.5290907, 0.1980252, 0.1012210)$, $f_{12} = (0.5800074, 0.8072991, 0.1645741)$.

The weights in the objective are
$\mathbf{w} = (0.1093683, 0.2339055, 0.5295364, 0.2302270, 0.4429267, 0.3831922, 0.1667756, 0.7351673, 0.1278835, 0.4373618)$.

The p for the ellipsoidal norms are
$p = (2.1, 2.3, 1.5, 3.1, 1.9, 1.9, 1.9, 1.9, 1.9, 1.7, 3.3, 1.9)$.

The linear transformation matrices for the ellipsoidal norms are:

$$M_1 = \begin{bmatrix} 0.3692412 & 0.7397360 & 0.3671468 \\ 0.3451773 & 0.5917062 & 0.7667686 \\ 0.7153005 & 0.3233307 & 0.5691886 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.5969909 & 0.1652245 & 0.9320327 \\ 0.1077514 & 0.0023804 & 0.9944936 \\ 0.9073230 & 0.5631396 & 0.9684730 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0.6143491 & 0.1460124 & 0.7866041 \\ 0.0613802 & 0.7311101 & 0.6258767 \\ 0.5635103 & 0.9968714 & 0.8857684 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0.4259558 & 0.1795297 & 0.3595944 \\ 0.0765066 & 0.1897343 & 0.1458442 \\ 0.4611495 & 0.7927964 & 0.3008546 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 0.8462288 & 0.3543530 & 0.9533479 \\ 0.8034236 & 0.5255093 & 0.0520969 \\ 0.0296419 & 0.9122241 & 0.2724550 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 0.6154476 & 0.0826227 & 0.3935258 \\ 0.8348387 & 0.2062091 & 0.8198316 \\ 0.3425691 & 0.6444498 & 0.2284179 \end{bmatrix}$$

$$M_7 = \begin{bmatrix} 0.9841331 & 0.6252488 & 0.3162361 \\ 0.9204588 & 0.0062540 & 0.2861219 \\ 0.8010166 & 0.2107349 & 0.6545077 \end{bmatrix}$$

$$M_8 = \begin{bmatrix} 0.5306548 & 0.3486466 & 0.8227931 \\ 0.7473973 & 0.7774893 & 0.1455634 \\ 0.2803321 & 0.3570228 & 0.5587755 \end{bmatrix}$$

$$M_9 = \begin{bmatrix} 0.1136328 & 0.6820894 & 0.9157958 \\ 0.7598759 & 0.9467710 & 0.9630559 \\ 0.9203295 & 0.4310774 & 0.8603421 \end{bmatrix}$$

$$M_{10} = \begin{bmatrix} 0.1660018 & 0.3996167 & 0.1370780 \\ 0.5331381 & 0.8280759 & 0.8089569 \\ 0.6531517 & 0.2877654 & 0.2869363 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 0.8655222 & 0.8695969 & 0.2533491 \\ 0.7964027 & 0.0003685 & 0.8457027 \\ 0.1553411 & 0.6526919 & 0.8278037 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 0.8813783 & 0.2787139 & 0.8239571 \\ 0.9325132 & 0.1975635 & 0.4511900 \\ 0.5321329 & 0.7430964 & 0.5640829 \end{bmatrix} .$$

The coefficients for the linear constraints are

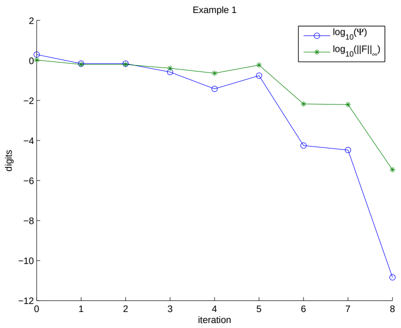
$$A = \begin{pmatrix} 0.0482753 & 0.4878658 & 0.0619276 \\ -0.8904399 & -0.8477093 & -0.4979966 \\ 0.8629468 & 0.5048775 & 0.7739134 \end{pmatrix} ,$$

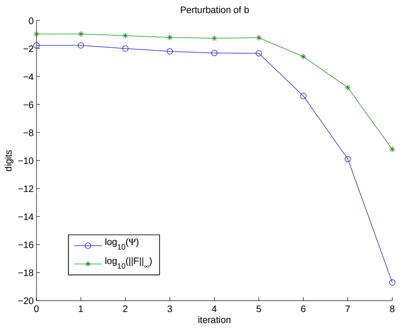
$$\mathbf{b} = (0.3484993, -1.0367463, 1.0501067)^T .$$

The minimal distances from the essential sites are

$$\mathbf{r} = (0.7952982, 0.1178979)^T .$$

We start from a zero solution: $\mathbf{x} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$, $\boldsymbol{\eta} = \mathbf{0}$. The iterates are summarized in the figure “Example 1”. In the figure, x-axis represents the iteration number. The blue plot depicts $log_{10}(\Psi)$. The green plot depicts $\log_{10}(\|F\|_\infty)$.





Assume some data need to be modified, due to some previous measure error or the availability of more advanced measuring instruments. We then use the old solution as the starting point to solve the new instances by Newton’s method, since Newton’s method has locally Q-quadratic convergence rate.

We perturb $\mathbf{b}$ by some random number in $(-0.5, 0.5)$ to

$$\mathbf{b} = (0.4528631, -0.9323825, 1.1544705)^T .$$

Then we use the perturbed nonmonotone Newton’s method to solve the problem with starting point being the solution to ‘Example 1’. The iterations are summarized in the figure “Perturbation of b”.

We also randomly perturb each element of A in the range $(-0.5, 0.5)$, to

$$A = \begin{pmatrix} -0.0613061 & 0.3782844 & -0.0476538 \\ -1.0000212 & -0.9572907 & -0.6075779 \\ 0.7533654 & 0.3952961 & 0.6643320 \end{pmatrix} \; ;$$

perturb each weight w in the range $(-0.5, 0.5)$ to

$\mathbf{w} = (0.7894956, 0.9140328, 1.2096637, 0.9103542, 1.1230539, 1.0633194, 0.8469029, 1.4152946, 0.8080108, 1.1174891)$;

perturb each existing sites $\mathbf{f}$ randomly in the range $(-0.5, 0.5)$, to
$f_1 = (0.7483169, 0.8840173, 0.3468729)^T$ $f_2 = (0.4161400, 0.3885744, 0.6530967)^T$ $f_3 = (0.9018664, 0.3132102, 0.5935753)^T$
$f_4 = (0.8224637, 0.4207312, 0.8078842)^T$ $f_5 = (0.3556989, 0.2451229, -0.0193689)^T$ $f_6 = (0.5094533, 0.0796051, 0.1080387$
$f_7 = (0.6317674, 0.6583052, 0.4611324)^T$ $f_8 = (0.3894556, 0.9110042, 0.5075373)^T$ $f_9 = (0.3897792, 0.0059204, 0.4002447)^T$
$f_{10} = (0.0644359, 0.1743426, 0.5966293)^T$ $f_{11} = (0.4422379, 0.1111724, 0.0143683)^T$ $f_{12} = (0.4931546, 0.7204463, 0.077721$

perturb each element of the linear matrices for the ellipsoidal norm randomly by a number in $(-0.5, 0.5)$ to

$$M_1 = \begin{bmatrix} 0.4750002 & 0.8454950 & 0.4729058 \\ 0.4509363 & 0.6974652 & 0.8725276 \\ 0.8210595 & 0.4290897 & 0.6749476 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.7027499 & 0.2709835 & 1.0377917 \\ 0.2135104 & 0.1081394 & 1.1002526 \\ 1.0130820 & 0.6688986 & 1.0742320 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0.7201081 & 0.2517714 & 0.8923631 \\ 0.1671392 & 0.8368691 & 0.7316357 \\ 0.6692693 & 1.1026304 & 0.9915274 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0.5317148 & 0.2852887 & 0.4653534 \\ 0.1822656 & 0.2954933 & 0.2516032 \\ 0.5669084 & 0.8985554 & 0.4066136 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 0.9519878 & 0.4601120 & 1.0591069 \\ 0.9091826 & 0.6312682 & 0.1578559 \\ 0.1354009 & 1.0179831 & 0.3782140 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 0.7212066 & 0.1883817 & 0.4992848 \\ 0.9405977 & 0.3119681 & 0.9255906 \\ 0.4483281 & 0.7502088 & 0.3341769 \end{bmatrix}$$

$$M_7 = \begin{bmatrix} 1.0898921 & 0.7310078 & 0.4219951 \\ 1.0262178 & 0.1120130 & 0.3918809 \\ 0.9067756 & 0.3164939 & 0.7602667 \end{bmatrix}$$

$$M_8 = \begin{bmatrix} 0.6364138 & 0.4544056 & 0.9285521 \\ 0.8531563 & 0.8832483 & 0.2513224 \\ 0.3860911 & 0.4627818 & 0.6645345 \end{bmatrix}$$

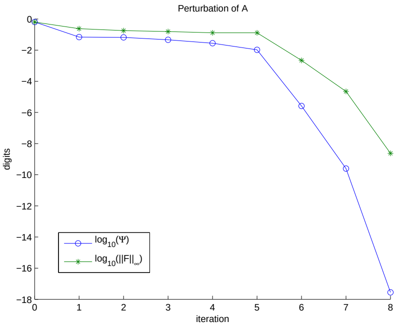
$$M_9 = \begin{bmatrix} 0.2193918 & 0.7878484 & 1.0215548 \\ 0.8656349 & 1.0525300 & 1.0688149 \\ 1.0260885 & 0.5368364 & 0.9661011 \end{bmatrix}$$

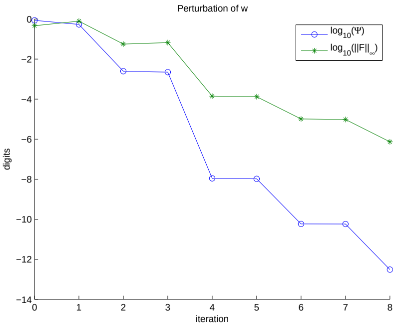
$$M_{10} = \begin{bmatrix} 0.2717608 & 0.5053757 & 0.2428370 \\ 0.6388970 & 0.9338349 & 0.9147159 \\ 0.7589107 & 0.3935244 & 0.3926953 \end{bmatrix}$$

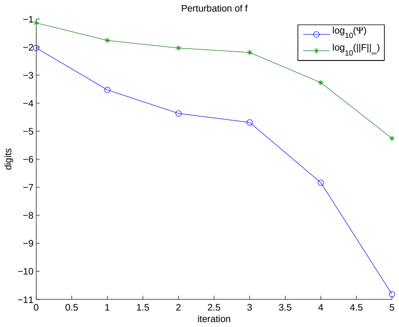
$$M_{11} = \begin{bmatrix} 0.9712812 & 0.9753558 & 0.3591081 \\ 0.9021617 & 0.1061274 & 0.9514617 \\ 0.2611001 & 0.7584509 & 0.9335627 \end{bmatrix}$$

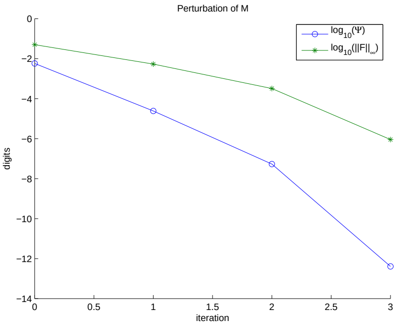
$$M_{12} = \begin{bmatrix} 0.9871373 & 0.3844729 & 0.9297161 \\ 1.0382722 & 0.3033225 & 0.5569490 \\ 0.6378919 & 0.8488554 & 0.6698419 \end{bmatrix} .$$

These instances are then solved by the perturbed nonmonotone Newton's method starting from the solution to 'Example 1'. The iterations are summarized in figures "Perturbation of A", "Perturbation of w", "Perturbation of f", and "Perturbation of M" respectively.









The above instances show the Q-quadratic convergence rate of the Newton's method. To find a global solution, randomly restarting of Newton's method can be used.

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