

Tadatoshi Miyamoto

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### Abstract

The Strong Reflection Principle (SRP) negates the continuum hypothesis (CH). To iteratively force SRP, we may use notions of forcing which add no new countable sequences of ordinals. Hence the iteration must add new reals at limit stages. We explicate this phenomenon with an example.

### Introduction

It is known that we may in some cases iterate  $\sigma$ -Baire (i.e. adds no new countable sequences of ordinals) proper or semiproper notions of forcing so that iterations themselves are  $\sigma$ -Baire. Examples include iterated forcing constructions of  $\text{Con}(\text{CH} + \neg\Diamond)$  or  $\text{Con}(\text{CH} + \text{the Cofinal bounding})$  due to [S] and [L-S].

On the other hand, we may iteratively force with notions of forcing which are  $\sigma$ -Baire and semiproper (p. 57 of [B]) to get the Strong Reflection Principle (SRP). However, SRP implies  $2^\omega = 2^{\omega_1} = \omega_2$  by [W]. Hence, this type of iteration must introduce new reals at limit stages. We ask how exactly this occurs ?

More closely, SRP implies  $\psi_{\text{AC}}$  and  $\psi_{\text{AC}}$  in turn implies  $2^\omega = 2^{\omega_1} = \omega_2$ . An alternative proof to this via the Weak Diamond of [D-S] is noted in Remark 3.52 on p. 109 of [W]. For an explicit use of the Weak Diamond, we may see, say, [T].

We pick up a crucial fact behind [D-S] on directed systems and their limits from, say, p. 940-951 of [S]. We use this well-known fact on direct limits to see how new reals are created in the iterated forcing construction of SRP.

To this effect, we tentatively formulate a stronger form of  $\psi_{\text{AC}}$ , denoted by  $\psi_{\text{AC}}^*$ . It is not hard to see that SRP implies  $\psi_{\text{AC}}^*$ . It appears an iterated forcing construction of  $\psi_{\text{AC}}^*$  gives a direct explication on how new reals are created at limit stages. The underlining basic idea appears to be very much similar to pp. 122-125 of [D-J] where Souslin trees are iterated.

More specifically, we give an example of an  $\omega$ -stage countable support iterated forcing  $P_\omega$ , assuming CH and  $\omega$ -many measurable cardinals exist. This  $P_\omega$  is gotten by iteratively forcing with a type of  $\sigma$ -Baire, semiproper notions of forcing and should be regarded as a typical intermediate interval of stages in the iterated forcing construction of  $\psi_{\text{AC}}^*$ . Among others, we have

- $P_\omega$  is semiproper whose generic object is a real.
- $P_\omega$  collapses many cardinals.
- $P_\omega$  is  ${}^\omega\omega$ -bounding.
- Hence this new real introduces no Cohen reals.

The second last fact may follow from a more general fact. By [S], proper together with  ${}^\omega\omega$ -bounding under countable support iterates. We may generalize this assuming CH so that semiproper together with  ${}^\omega\omega$ -bounding under a type of revised countable support iterates ([M1]). This revised countable support of [M] takes the inverse limit at limit stages  $\alpha$  with  $\text{cf}(\alpha) = \omega$ .

### §1. Preliminary

We set our terminologies such as directed systems and recall a crucial fact on their limits, referred to as direct limits.

**1.1 Definition.** Let  $\gamma$  be an ordinal with  $\omega_1 < \gamma < \omega_2$ . We say  $\langle X_i \mid i < \omega_1 \rangle$  is a *continuously increasing sequence of countable subsets of  $\gamma$  which converges to  $\gamma$* , simply denoted by  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$ , if

- Each  $X_i$  is a countable subset of  $\gamma$ ,
- If  $i < j < \omega_1$ , then  $X_i \subseteq X_j$  (equality allowed),
- If  $j$  is a limit ordinal with  $j < \omega_1$ , then  $X_j = \bigcup \{X_i \mid i < j\}$ ,
- $\gamma = \bigcup \{X_i \mid i < \omega_1\}$ .

**1.2 Definition.** We say  $(\langle \theta_i \mid i \leq \omega_1 \rangle, \langle f_{ij} \mid i < j \leq \omega_1 \rangle)$  is a *directed system*, if

- If  $i < \omega_1$ , then  $\theta_i < \omega_1$  and  $\omega_1 < \theta_{\omega_1} < \omega_2$ ,
- Each  $f_{ij} : (\theta_i, <) \longrightarrow (\theta_j, <)$  is an order-preserving map,
- If  $i < j < k \leq \omega_1$ , then  $f_{ik} = f_{jk} \circ f_{ij}$  (commutative),
- If  $j$  is a limit ordinal with  $j \leq \omega_1$ , then  $\theta_j = \{f_{ij}(x) \mid x \in \theta_i, i < j\}$ .

For  $\alpha < \omega_1$ , we similarly say  $(\langle \theta_i \mid i \leq \alpha \rangle, \langle f_{ij} \mid i < j \leq \alpha \rangle)$  is a directed system, with obvious modifications. Here is an important basic relation with  $\alpha = \omega_1$ . We omit its proof.

**1.3 Proposition.** Let  $\gamma$  be an ordinal with  $\omega_1 < \gamma < \omega_2$ .

- (1) For each  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$ , there exists a unique directed system  $(\langle \theta_i \mid i \leq \omega_1 \rangle, \langle f_{ij} \mid i < j \leq \omega_1 \rangle)$  such that for all  $i < \omega_1$ , we have  $X_i = \{f_{i\omega_1}(x) \mid x \in \theta_i\}$  and  $\theta_{\omega_1} = \gamma$ .
- (2) For each directed system  $(\langle \theta_i \mid i \leq \omega_1 \rangle, \langle f_{ij} \mid i < j \leq \omega_1 \rangle)$  with  $\theta_{\omega_1} = \gamma$ , there exists a unique  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$  such that for all  $i < \omega_1$ , we have  $X_i = \{f_{i\omega_1}(x) \mid x \in \theta_i\}$ .
- (3) These two correspondances are mutually inverses.

Hence we may freely switch between  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$  and directed systems with  $\theta_{\omega_1} = \gamma$ .

Now we recall a crucial fact (existence and uniqueness) on the direct limit.

**1.4 Lemma.** Let  $\alpha$  be a limit ordinal with  $\alpha \leq \omega_1$  and  $DS = (\langle \theta_i \mid i \leq \alpha \rangle, \langle f_{ij} \mid i < j \leq \alpha \rangle)$  be a directed system. Then there exists a unique (up to an isomorphism)  $((D, E), \langle F_i \mid i < \alpha \rangle)$  such that

- (1)  $(D, E)$  is a linear order,
- (2) Each  $F_i : (\theta_i, <) \longrightarrow (D, E)$  is an order-preserving map,
- (3) If  $i < j < \alpha$ , then  $F_i = F_j \circ f_{ij}$  (commutative),
- (4)  $D = \{F_i(x) \mid x \in \theta_i, i < \alpha\}$ .

From (1) and (2), we may imply

- Each  $F_i : (\theta_i, <) \longrightarrow (D, E)$  is a bi-order-preserving map,

From this, (3) and (4), we may imply

- $E = \{(F_i(x_1), F_i(x_2)) \mid x_1, x_2 \in \theta_i \text{ with } x_1 < x_2, i < \alpha\}$ .

We refer to these  $((D, E), \langle F_i \mid i < \alpha \rangle)$  as the *direct limit* of the directed system  $DS$ . In general,  $(D, E)$  may or may not be well-founded. If  $(D, E)$  is well-founded, we opt to choose its order-type  $(\theta, <)$  together with corresponding maps  $F_i : (\theta_i, <) \longrightarrow (\theta, <)$  as the unique direct limit of the system  $DS$ .

*Proof.* Since everything here is routine, we just mention main points.

(Existence): Let  $D_0 = \{(i, x) \mid x \in \theta_i, i < \alpha\}$  and introduce an equivalence relation  $\sim$  on  $D_0$  as follows;

$$(i_1, x_1) \sim (i_2, x_2),$$

if there exists  $k > i_1, i_2$  such that  $f_{i_1 k}(x_1) = f_{i_2 k}(x_2)$ .

We denote the equivalence class of  $(i, x)$  by  $\overline{(i, x)}$  and set

$$D = \{\overline{(i, x)} \mid (i, x) \in D_0\}.$$

For  $\overline{(i_1, x_1)}, \overline{(i_2, x_2)} \in D$ , we may well-define a linear order (irreflexive and transitive) relation  $E$  on  $D$  as follows;

$$\overline{(i_1, x_1)} E \overline{(i_2, x_2)},$$

if there exists  $k > i_1, i_2$  such that  $f_{i_1 k}(x_1) < f_{i_2 k}(x_2)$ .

For each  $i < \alpha$ , define  $F_i : (\theta_i, <) \longrightarrow (D, E)$  as follows;

$$F_i(x) = \overline{(i, x)}.$$

Then these  $((D, E), \langle F_i \mid i < \alpha \rangle)$  work.

(Uniqueness up to isomorphisms): Let  $((D_1, E_1), \langle F_i^1 \mid i < \alpha \rangle)$  and  $((D_2, E_2), \langle F_i^2 \mid i < \alpha \rangle)$  be two possible limits. Then we may well-define a map  $\phi : (D_1, E_1) \longrightarrow (D_2, E_2)$  by

$$\phi(F_i^1(x)) = F_i^2(x).$$

It is routine to check that this is an isomorphism such that for all  $i < \alpha$ , we have  $\phi \circ F_i^1 = F_i^2$ .

□

## §2. Single step

Let  $\langle S_i \mid i < \omega_1 \rangle$  be a sequence of stationary subsets of  $\omega_1$ . If  $\kappa$  is a measurable cardinal, then there exists a notion of forcing  $P$  which is semiproper,  $\sigma$ -Baire (i.e., adds now new sequences of ordinals of length  $\omega$ ) and adds  $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$  such that for all  $i < \omega_1$ , we have  $\text{o.t.}(X_i) \in S_i$ , where  $\text{o.t.}(X_i)$  denotes the order-type of  $X_i$ .

**2.1 Definition.** Let  $\langle S_i \mid i < \omega_1 \rangle$  be a sequence of stationary subsets of  $\omega_1$  and  $\kappa$  be a measurable cardinal. We define a notion of forcing  $P = P(\langle S_i \mid i < \omega_1 \rangle, \kappa)$  as follows;

$$p = \langle X_i^p \mid i \leq \alpha^p \rangle \in P,$$

if the following are satisfied

- $\alpha^p < \omega_1$ ,
- Each  $X_i^p$  is a countable subset of  $\kappa$  with  $\text{o.t.}(X_i^p) \in S_i$ ,
- If  $i < j \leq \alpha^p$ , then  $X_i^p \subseteq X_j^p$  (increasing, equality allowed),
- If  $j$  is a limit ordinal with  $j \leq \alpha^p$ , then  $X_j^p = \bigcup \{X_i^p \mid i < j\}$  (continuous).

For  $p, q \in P$ , we set

$$q \leq p,$$

$$\text{if } p \subseteq q.$$

The relevant fact with the measurable cardinal  $\kappa$  is the following.

**2.2 Lemma.** Let  $\mathcal{U}$  be a normal measure and  $\chi$  be a regular cardinal with  $\chi \geq (2^\kappa)^+$  so that  $\mathcal{U} \in H_\chi$ . For any countable elementary substructure  $N$  of  $H_\chi$  with  $\mathcal{U} \in N$  and any  $s \in \bigcap (\mathcal{U} \cap N)$ , we may define

$$N(s) = \{f(s) \mid f \in N\}.$$

Then we have

- (1)  $\sup(N \cap \kappa) < s$ ,
- (2)  $\{s\} \cup N \subset N(s)$ ,  $N(s)$  is a countable elementary substructure of  $H_\chi$ ,
- (3) For all  $a \in N \cap H_\kappa$ , we have  $a \cap N = a \cap N(s)$ ,
- (4)  $s$  is the least ordinal in  $N(s) \setminus N$ .

□

By repeatedly applying the above we get

**2.3 Lemma.** Let  $\mathcal{U}$  and  $\chi$  be as above. For any countable elementary substructure  $N$  of  $H_\chi$  with  $\mathcal{U} \in N$ , there exists a sequence  $\langle N_i \mid i < \omega_1 \rangle$  of countable elementary substructures of  $H_\chi$  such that

- For all  $i < \omega_1$ ,  $N_i \cap \omega_1$  is constantly fixed,
- $\langle \text{o.t.}(N_i \cap \kappa) \mid i < \omega_1 \rangle$  is a club in  $\omega_1$ .

In particular, for any stationary subset  $S$  of  $\omega_1$ , we have  $M$  such that

- $M$  is a countable elementary substructure of  $H_\chi$  with  $N \subset M$ ,
- $M \cap \omega_1 = N \cap \omega_1$ ,
- $\text{o.t.}(M \cap \kappa) \in S$ .

We investigate properties of  $P$ . We begin with its density.

**2.4 Lemma.** For any  $p \in P$ ,  $\alpha^p < \alpha < \omega_1$  and  $\xi \in \kappa$ , there exists  $q \in P$  such that  $q \leq p$ ,  $\alpha^q = \alpha$  and  $\xi \in X_\alpha^q$ .

*Proof.* By induction on  $\alpha$ . We provide some details when  $\alpha$  is limit. Let  $p, \xi$  be given. Let  $\langle \alpha_n \mid n < \omega \rangle$  be a strictly increasing sequence of countable ordinals such that  $\alpha_0 = \alpha^p$  and  $\sup\{\alpha_n \mid n < \omega\} = \alpha$ . Fix a sufficiently large regular cardinal  $\chi$  and a countable elementary substructure  $N$  of  $H_\chi$  with relevant parameters in it. By 2.3 Lemma, we may assume

$$\text{o.t.}(N \cap \kappa) \in S_\alpha.$$

Let  $\langle \xi_n \mid n < \omega \rangle$  enumerate  $N \cap \kappa$ . By repeatedly applying induction hypothesis, we may construct  $\langle q_n \mid n < \omega \rangle$  such that

- $q_0 = p$ ,
- $q_n \leq p$ ,  $q_n \in N$  and  $\alpha^{q_n} = \alpha_n$ ,
- $q_{n+1} \leq q_n$ ,  $\xi_n \in X_{\alpha_{n+1}}^{q_{n+1}}$ .

Let  $q = \bigcup\{q_n \mid n < \omega\} \cup \{(\alpha, N \cap \kappa)\}$ . Then this  $q$  works.

□

**2.5 Lemma.**  $P$  is semiproper and  $\sigma$ -Baire.

*Proof.* Let  $\chi$  be a sufficiently large regular cardinal and  $N$  be a countable elementary substructure of  $H_\chi$  with  $P \in N$ . Then we have  $\kappa \in N$  and so may fix a normal measure  $\mathcal{U} \in N$ . Let  $p \in P \cap N$ . Take a countable elementary substructure  $M$  of  $H_\chi$  such that  $N \subset M$ ,  $N \cap \omega_1 = M \cap \omega_1$  and  $\text{o.t.}(M \cap \kappa) \in S_{N \cap \omega_1}$ . Let  $\langle p_n \mid n < \omega \rangle$  be any  $(P, M)$ -generic sequence. Then  $\sup\{\alpha^{p_n} \mid n < \omega\} = M \cap \omega_1$  and  $\bigcup\{X_{\alpha^{p_n}}^{p_n} \mid n < \omega\} = M \cap \kappa$  hold. Hence  $q = \bigcup\{p_n \mid n < \omega\} \cup \{(N \cap \omega_1, M \cap \kappa)\}$  is a  $(P, M)$ -generic condition. Thus we have  $q \Vdash_P "M[\dot{G}] \cap \chi = M \cap \chi"$  and so  $q \Vdash_P "N[\dot{G}] \cap \omega_1 \subseteq M[\dot{G}] \cap \omega_1 = M \cap \omega_1 = N \cap \omega_1"$ . Therefore  $q \Vdash_P "N[\dot{G}] \cap \omega_1 = N \cap \omega_1"$ .

Since  $q$  is a lower bound of a  $(P, M)$ -generic sequence,  $P$  is  $\sigma$ -Baire.

□

**2.6 Lemma.**  $P$  is of size  $\kappa$  and so has the  $\kappa^+$ -c.c.

**2.7 Lemma.** Let  $G$  be a  $P$ -generic filter over the ground model  $V$ . Then we may set  $\bigcup G = \langle X_i \mid i < \omega_1 \rangle$  so that for all  $i < \omega_1$ , we have  $\text{o.t.}(X_i) \in S_i$ . This  $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$  induces its equivalent directed system  $(\langle \theta_i \mid i \leq \omega_1 \rangle, \langle f_{ij} \mid i < j \leq \omega_1 \rangle)$  with  $\theta_{\omega_1} = \kappa$ . In particular,  $|\kappa| = \omega_1$  and for all  $i < \omega_1$ , we have  $\theta_i \in S_i$ .

*Proof.* By genericity. □

**2.8 Lemma.** In the generic extensions of  $P$ , the continuum hypothesis (CH) holds.

*Proof.* For each  $\alpha < \omega_1$ , let  $r_\alpha = \{n < \omega \mid \text{o.t.}(X_{\alpha+n}) \text{ is a successor point in } S_{\alpha+n}\}$ . Then by genericity,  $\langle r_\alpha \mid \alpha < \omega_1 \rangle$  enumerates every reals in the extension. □

### §3. $\omega$ -stage iteration

It is possible to collapse many cardinals while preserving  $\omega_1$  by adding a real. Notice that the following iteration also shows  $\sigma$ -Baireness may not iterate under CS-iterations. ([DJ])

**3.1 Theorem.** (CH) Let  $\langle \kappa_n \mid n < \omega \rangle$  be a strictly increasing sequence of measurable cardinals. Then we have a countable support iterated forcing  $(\langle P_n \mid n \leq \omega \rangle, \langle \dot{Q}_n \mid n < \omega \rangle)$  such that

- (1)  $P_0 = \{\emptyset\}$ ,
- (2)  $P_n \in H_{\kappa_n}$  is semiproper and  $\sigma$ -Baire,
- (3)  $\dot{Q}_n$  is a  $P_n$ -name such that  $\Vdash_{P_n} \text{“}\dot{Q}_n \text{ is semiproper and } \sigma\text{-Baire”}$ ,
- (4)  $P_{n+1} = P_n * \dot{Q}_n$  and  $\Vdash_{P_{n+1}} \text{“}|\kappa_n| = \omega_1\text{”}$ ,
- (5)  $P_\omega$  is semiproper and so preserves  $\omega_1$ ,
- (6) There exists a  $P_\omega$ -name  $\dot{r}$  such that  $\Vdash_{P_\omega} \text{“}\dot{r} \text{ is a real, } \dot{G}_\omega \in V[\dot{r}] \text{ and so } \dot{r} \notin V\text{”}$ , where  $\dot{G}_\omega$  denotes the canonical  $P_\omega$ -name of the  $P_\omega$ -generic filters over  $V$ .

*Proof.* We first out-line and then provide more details.

(Out-line) Since we assume CH, we may fix  $h : \omega_1 \rightarrow H_{\omega_1}$  so that for all  $a \in H_{\omega_1}$ , the following

$$S_a = \{i < \omega_1 \mid h(i) = a\}$$

are stationary in  $\omega_1$ .

Notice that above is upward-absolute as long as we force with notions of forcing which are semiproper and  $\sigma$ -Baire.

We first force  $\langle X_i^0 \mid i < \omega_1 \rangle \nearrow k_0$  over  $V$ . We do not require anything specific on the  $\text{o.t.}(X_i^0)$ 's in  $V[G_1]$ , though we could.

Suppose we have gotten  $\langle X_i^n \mid i < \omega_1 \rangle \nearrow \kappa_n$  in  $V[G_{n+1}]$ . This is turned into a directed system

$$DS_n = (\langle \theta_i^n \mid i \leq \omega_1 \rangle, \langle f_{ij}^n \mid i < j \leq \omega_1 \rangle)$$

with

$$\theta_{\omega_1}^n = \kappa_n.$$

Each  $S_{(\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)}$  remains stationary and so does  $\kappa_{n+1}$  measurable. Hence we may further force over  $V[G_{n+1}]$  with

$$P(\langle S_{(\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)} \mid i < \omega_1 \rangle, \kappa_{n+1})$$

so that we have  $\langle X_i^{n+1} \mid i < \omega_1 \rangle \nearrow k_{n+1}$  in  $V[G_{n+2}]$ . This is turned into a directed system

$$DS_{n+1} = (\langle \theta_i^{n+1} \mid i \leq \omega_1 \rangle, \langle f_{ij}^{n+1} \mid i < j \leq \omega_1 \rangle)$$

such that for all  $i < \omega_1$

$$h(\theta_i^{n+1}) = (\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n),$$

and

$$\theta_{\omega_1}^{n+1} = \kappa_{n+1}.$$

So  $\theta_i^{n+1}$  recovers  $(\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)$  via  $h$ . For  $\alpha \leq \omega_1$ , let

$$DS_n[\alpha] = (\langle \theta_i^n \mid i \leq \alpha \rangle, \langle f_{ij}^n \mid i < j \leq \alpha \rangle).$$

We may calculate  $\langle DS_n[0] \mid n < \omega \rangle$ ,  $\langle DS_n[1] \mid n < \omega \rangle$ ,  $\langle DS_n[2] \mid n < \omega \rangle$ ,  $\dots$  via  $r = \langle \theta_0^n \mid n < \omega \rangle$  and  $h$  in turn. Since direct limits are unique, we may recover  $\langle DS_n[\omega] \mid n < \omega \rangle$  and so  $\langle DS_n[\omega + 1] \mid n < \omega \rangle$  by

$$h(\theta_\omega^{n+1}) = (\theta_\omega^n, f_{\omega\omega+1}^n, \theta_{\omega+1}^n).$$

This way we may recover

$$\langle DS_n[\alpha] \mid n < \omega \rangle$$

from  $r$  and  $h$  by recursion on  $\alpha$ . We may regard  $r \in {}^\omega \omega_1$  is a real and that  $G_\omega$  is  $\langle DS_n \mid n < \omega \rangle$ . Hence  $G_\omega \in V[r]$ .

(Details) We construct a countable support iterated forcing

$$(\langle P_n \mid n \leq \omega \rangle, \langle \dot{Q}_n \mid n < \omega \rangle)$$

by recursion on  $n \leq \omega$ .

First let  $P_0 = \{\emptyset\}$  and  $P_1 = P_0 * \dot{Q}_0$ , where  $\dot{Q}_0$  is a  $P_0$ -name such that

- $\Vdash_{P_0} \dot{Q}_0 = P(\langle \omega_1 \mid i < \omega_1 \rangle, \kappa_0)$ .

Then in  $V^{P_1}$ , we have the canonical  $\langle X_i^0 \mid i < \omega_1 \rangle \nearrow \kappa_0$ . Notice we require nothing specific on the o.t.  $(X_i^0)$ 's, though we could. This is turned into a canonical directed system

- $DS_0 = (\langle \theta_i^0 \mid i \leq \omega_1 \rangle, \langle f_{ij}^0 \mid i < j \leq \omega_1 \rangle)$ .

Suppose we have constructed  $P_{n+1}$  such that

- $P_{n+1} \in H_{\kappa_{n+1}}$ ,  $P_{n+1}$  is semiproper and  $\sigma$ -Baire.
- $\Vdash_{P_{n+1}} DS_n = (\langle \theta_i^n \mid i \leq \omega_1 \rangle, \langle f_{ij}^n \mid i < j \leq \omega_1 \rangle)$  is the canonical directed system with  $\theta_{\omega_1}^n = \kappa_n$

Then in  $V^{P_{n+1}}$ , for all  $i < \omega_1$ ,  $S_{(\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)}$  remain stationary and so does  $\kappa_{n+1}$  measurable. Let  $\dot{Q}_{n+1}$  be a  $P_{n+1}$ -name such that

- $\Vdash_{P_{n+1}} \dot{Q}_{n+1} = P(\langle S_{(\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)} \mid i < \omega_1 \rangle, \kappa_{n+1})$ .

Let  $P_{n+2} = P_{n+1} * \dot{Q}_{n+1}$ . Then  $P_{n+2}$  is semiproper and  $\sigma$ -Baire. In  $V^{P_{n+2}}$ , we have the canonical directed system  $DS_{n+1} = (\langle \theta_i^{n+1} \mid i \leq \omega_1 \rangle, \langle f_{ij}^{n+1} \mid i < j \leq \omega_1 \rangle)$  such that

- For all  $i < \omega_1$ ,  $h(\theta_i^{n+1}) = (\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)$ .

This completes the construction of  $P_n$  for all  $n < \omega$ . Let  $P_\omega$  be the inverse limit of the  $P_n$ 's. Then by, say, [M]

- $P_\omega$  is semiproper and so preserves  $\omega_1$ .

Now we show this iteration works. To this end let  $G_\omega$  be  $P_\omega$ -generic filter over  $V$ . Let

$$DS_n = (\langle \theta_i^n \mid i \leq \omega_1 \rangle, \langle f_{ij}^n \mid i < j \leq \omega_1 \rangle)$$

be the corresponding canonical directed systems so that for all  $i < \omega_1$  and all  $n < \omega$

$$h(\theta_i^{n+1}) = (\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)$$

and

$$\theta_{\omega_1}^n = \kappa_n.$$

We want to recover  $\langle DS_n \mid n < \omega \rangle$  via  $r = \langle \theta_0^n \mid n < \omega \rangle$  and  $h$ . So we recursively construct (omitting index  $\alpha$  in  $\phi_i^n$  and  $g_{ij}^n$  with no confusion)

$$\langle n \mapsto (\langle \phi_i^n \mid i \leq \alpha \rangle, \langle g_{ij}^n \mid i < j \leq \alpha \rangle) \mid n < \omega \rangle$$

via  $r$  and  $h$  by recursion on  $\alpha$  and show for all  $n < \omega$

$$\langle \phi_i^n \mid i \leq \alpha \rangle = \langle \theta_i^n \mid i \leq \alpha \rangle, \langle g_{ij}^n \mid i < j \leq \alpha \rangle = \langle f_{ij}^n \mid i < j \leq \alpha \rangle$$

by induction on  $\alpha$ .

Our intended recursive construction is as follows; For all  $n < \omega$ , let

$$\phi_0^n = r(n).$$

Suppose we have constructed

$$\langle n \mapsto (\langle \phi_i^n \mid i \leq \alpha \rangle, \langle g_{ij}^n \mid i < j \leq \alpha \rangle) \mid n < \omega \rangle.$$

We want  $\langle \phi_{\alpha+1}^n \mid n < \omega \rangle$  and  $\langle g_{i\alpha+1}^n \mid i \leq \alpha \rangle$ . Calculate  $h(\phi_\alpha^{n+1})$ . If  $h(\phi_\alpha^{n+1}) = (\phi_\alpha^n, g, \phi)$  for some ordinal  $\phi$  and a map  $g : \phi_\alpha^n \rightarrow \phi$ , then let

$$\phi_{\alpha+1}^n = \phi, g_{\alpha\alpha+1}^n = g$$

and for each  $i < \alpha$ , let

$$g_{i\alpha+1}^n = g \circ g_{i\alpha}^n.$$

Otherwise, we do not care.

Next, suppose  $\alpha \leq \omega_1$  is a limit ordinal and have constructed

$$\langle n \mapsto (\langle \phi_i^n \mid i < \alpha \rangle, \langle g_{ij}^n \mid i < j < \alpha \rangle) \mid n < \omega \rangle.$$

If all of these are directed systems and their direct limits are well-founded, then let  $(\phi_\alpha^n, <)$  be the corresponding order-types and  $\langle g_{i\alpha}^n \mid i < \alpha \rangle$  be the corresponding maps from  $\phi_i^n$  into  $\phi_\alpha^n$ .

Otherwise, we do not care.

This completes the construction of  $\phi_i^n$ 's and  $g_{ij}^n$ 's. Now it is straightforward to show for all  $n < \omega$

$$\langle \phi_i^n \mid i \leq \alpha \rangle = \langle \theta_i^n \mid i \leq \alpha \rangle, \langle g_{ij}^n \mid i < j \leq \alpha \rangle = \langle f_{ij}^n \mid i < j \leq \alpha \rangle$$

by induction on  $\alpha$ .

In particular,  $\langle DS_n \mid n < \omega \rangle \in V[r]$  and so  $V[G_\omega] = V[r]$  holds.

## §4. How reals are added

It has been known ([W]) that the Strong Reflection Principle (SRP) implies  $2^\omega = 2^{\omega_1} = \omega_2$ . However, the relevant notions of forcing are  $\sigma$ -Baire. This means that we must add lots of new reals at limit stages, when we force SRP. Our question is how those new reals were created while we iterate? To challenge this, we tentatively formulate the following.

**4.1 Definition.**  $\psi_{AC}^*$ , if for any sequence  $\langle S_i \mid i < \omega_1 \rangle$  of stationary subsets of  $\omega_1$ , there exists  $\gamma$  with  $\omega_1 < \gamma < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$  such that for all  $i < \omega_1$ ,

$$\text{o.t.}(X_i) \in S_i.$$

By [W], SRP implies  $\psi_{AC}$  and  $\psi_{AC}$  in turn implies  $2^\omega = 2^{\omega_1} = \omega_2$ . We have

**4.2 Proposition.** (1) SRP implies  $\psi_{AC}^*$ ,

(2)  $\psi_{AC}^*$  implies  $\psi_{AC}$ .

We may force  $\psi_{AC}^*$  over the ground model  $V$  where there exists a regular cardinal  $\rho$  such that  $\{\kappa < \rho \mid \kappa \text{ is measurable}\}$  is cofinal below  $\rho$ . Hence by [D-D], [A-W] and [L-S], we have

**4.3 Proposition.** The following are all equiconsistent.

- $\text{Con}(\psi_{AC}^*)$ .
- $\text{Con}(\psi_{AC})$ .
- $\text{Con}(\text{the Cofinal bounding})$ .
- $\text{Con}(\text{There exists a regular cardinal below which measurable cardinals cofinally exist})$ .

We sort of abstractly describe a  $\rho$ -stage semiproper iterated generic extension where  $\psi_{AC}^*$  holds. We explicate how new reals are added. This observation is a reformulation of a direct proof to:  $\psi_{AC}^*$  implies  $2^\omega = 2^{\omega_1} = \omega_2$ .

**4.4 Proposition.** Let  $V[G_\alpha]$  ( $\alpha \leq \rho$ ) denote the intermediate stages when we force  $\psi_{AC}^*$ . Let  $V[G_\alpha]$  be a stage where CH holds and fix  $h : \omega_1 \rightarrow (H_{\omega_1})^{V[G_\alpha]}$  such that  $h \in V[G_\alpha]$ , for all  $a \in (H_{\omega_1})^{V[G_\alpha]}$ ,  $\{i < \omega_1 \mid h(i) = a\}$  is stationary. Then either (1) or (2) holds in  $V[G_\rho]$ .

- (1) We may trace generically added directed systems  $\langle n \mapsto (\langle \theta_i^n \mid i < \omega_1 \rangle, \langle f_{ij}^n \mid i < j \leq \omega_1 \rangle) \mid n < \omega \rangle$  such that for all  $n \in \omega$  and all  $i < \omega_1$ ,  $f_{ii+1}^n \in V[G_\alpha]$  and  $h(\theta_i^{n+1}) = (\theta_i^n, f_{ii+1}^n, \theta_{i+1}^n)$ . In this case,  $\langle \theta_0^n \mid n < \omega \rangle \notin V[G_\alpha]$  holds and this provides a new real over  $V[G_\alpha]$ .
- (2) There exists a directed system  $(\langle \theta_i \mid i < \omega_1 \rangle, \langle f_{ij} \mid i < j \leq \omega_1 \rangle)$  such that some  $f_{ii+1} \notin V[G_\alpha]$ . In this case,  $f_{ii+1}$  provides a new real over  $V[G_\alpha]$ . (This should be understood as tracing as in (1) failed at some  $n < \omega$ .)

Since there are cofinally many stages (e.g. every successor stages) where CH holds when we force  $\psi_{AC}^*$ , we must add cofinally many new reals over  $V$  in  $V[G_\rho]$ .

*Proof.* Argue as in §3. □

Since then, D. Aspero kindly e-mailed us that the following (1) is “No”. First collapse any cardinal  $\kappa \geq \omega_2$  by the countable conditions so that  $\kappa$  becomes of size  $\omega_1$ . Then almost disjoint code the induced new subset of  $\omega_1$  by a real. Hence it needs no large cardinals at all.

**4.5 Question.** (1) Can you need large cardinals to get a notion of forcing whose generic object is a new real, preserves  $\omega_1$  and collapses many cardinals?

(2) Can you separate  $\psi_{AC}^*$  of this note and  $\psi_{AC}$  ?

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Mathematics  
Nanzan University  
27 Seirei-cho, Seto-shi  
489-0863 Japan  
miyamoto@nanzan-u.ac.jp