Ladder Operators for q-orthogonal Polynomials [*]

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Abstract

The $q-$ difference analog of the classical ladder operators is derived for those orthogonal polynomials arising from a class of indeterminate moments problem.

1 Introduction

This work is a follow up to our work [5] where we derived raising and lowering operators for polynomials orthogonal with respect to absolutely continuous measures μ under certain smoothness assumptions of μ'. This approach goes back to [3], [4], and [13]. The raising and lowering operators derived in these references are are differential operators. It was later realized that a similar theory exists for polynomials orthogonal with respect to a measure with masses at the union of at most two geometric progressions, $\{aq^n, bq^n\}$, for some $q \in (0, 1)$, [9]. The corresponding theory for difference operators is in [11]. This material is reproduced in [10]. The raising and lowering operators involve two functions $A_n(x)$ and $B_n(x)$ which satisfy certain recurrence relations. In the case of differential operators we have demonstrated that the knowledge of $A_n(x)$ and $B_n(x)$ determines the polynomials uniquely in the cases of Hemite, Laguerre, and Jacobi polynomials, see [6]. This is done through recovering the properties of the polynomials including the three

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term recurrence relation which generates the polynomials. This work shows that the corresponding functions determines the polynomials in the cases of Stieltjes-Wigert and q-Laguerre polynomials.

The orthogonal polynomials which arise from indeterminate moment problems have discrete and absolutely continuous orthogonality measures [1]. In many instances it is more convenient to work with absolutely continuous measures [10, Chapter 21].

In this work we derive raising and lowering operators for polynomials orthogonal with respect to absolutely continuous measures. We shall assume that $\{P_n(x)\}$ are monic orthogonal polynomials, so that

$$\int_0^\infty w(x)P_m(x)P_n(x)dx = \zeta_n\delta_{m,n}. \tag{1.1}$$

A weight function w leads to a potential u defined by

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}, \tag{1.2}$$

where D_q is the q-difference operator

$$(D_q f)(x) = \frac{f(x)-f(qx)}{x-qx}. \tag{1.3}$$

Every monic sequence of orthogonal polynomials satisfies a three term recurrence relation of the form

$$(x-\alpha_n)P_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x). \tag{1.4}$$

We also write the monic polynomials $P_n(x)$ as follows:

$$P_n(x) = x^n + \mathsf{p}_1(n)x^{n-1} + \ldots$$

and it follows immediately from the three term recurrence relations ((1.4)) that

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1). \tag{1.5}$$

A main result of this work is the following theorem.

Theorem 1.1. *Let*

$$A_n(x) := \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}P_n(y)P_n(y/q)\,w(y)dy, \tag{1.6}$$

$$B_n(x) := \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}P_n(y)P_{n-1}(y/q)\,w(y)dy. \tag{1.7}$$

Then we have the lowering relation

$$D_q P_n(x) = \beta_n A_n(x) P_{n-1}(x) - B_n(x) P_n(x). \tag{1.8}$$

Theorem 1.1 will be proved in §2 along with the difference equations satisfied by $A_n(x)$ and $B_n(x)$;

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n) A_n(x) + x(q-1) \sum_{j=0}^{n} A_j(x) - u(qx), \tag{1.9}$$

$$1 + (x - \alpha_n) B_{n+1}(x) - (qx - \alpha_n) B_n(x) = \beta_{n+1} A_{n+1}(x) - \beta_n A_{n-1}(x). \tag{1.10}$$

The identities (1.9)–(1.10) will be referred to as the supplementary conditions.

Theorem 1.1 is the q-analogue of

$$P_n'(x) = \beta_n A_n(x) P_{n-1}(x) - B_n(x) P_n(x)$$

of [5].

Let

$$L_{1,n} := D_q + B_n(x), \tag{1.11}$$

be the lowering operator. Thus (1.8) is

$$L_{1,n} P_n(x) = \beta_n A_n(x) P_{n-1}(x). \tag{1.12}$$

In §3 we provide a raising operator $L_{2,n}$.

It is useful to recall the following analogue of the product rule

$$D_q(f(x)g(x)) = (D_q f(x)) g(x) + f(xq) D_q g(x). \tag{1.13}$$

The following lemma, whose proof is a calculus exercise, will be used in the proofs of our main results.

Lemma 1.2. *If the integrals*

$$\int_0^\infty f(x) g(x) \frac{dx}{x}, \quad \int_0^\infty f(x) g(qx) \frac{dx}{x},$$

exist then the following q-analogue of integration by parts holds

$$\int_0^\infty f(x) D_q g(x) dx = -\frac{1}{q} \int_0^\infty g(x) D_{q^{-1}} f(x) dx. \tag{1.14}$$

An immediate consequence of Lemma 1.2 and (1.1) is

$$\int_0^\infty u(y)P_n(y)P_n(y/q)w(y)dy = 0. \tag{1.15}$$

We also have

$$\int_0^\infty u(y)P_{n+1}(y)P_n(y/q)w(y)dy = \frac{1-q^{n+1}}{1-q}q\zeta_n, \tag{1.16}$$

which follows from (1.13), (1.2), (1.14), and the fact that

$$D_q x^n = \frac{1-q^n}{1-q}x^{n-1}.$$

2 Proofs

We shall need the fact [14], [10]

$$\zeta_n = \zeta_0\beta_1\beta_2\ldots\beta_n, \tag{2.1}$$

and the Christoffel–Darboux identity [14], [10]

$$\sum_{k=0}^{n-1}\frac{P_k(x)P_k(y)}{\zeta_k} = \frac{P_n(x)P_{n-1}(y)-P_n(y)P_{n-1}(x)}{\zeta_{n-1}(x-y)}. \tag{2.2}$$

Proof of Theorem 1.1. Let $D_qP_n(x) = \sum_{k=0}^{n-1} c_{n,k}P_k(x)$. Then

$$\zeta_k c_{n,k} = \int_0^\infty P_k(y)w(y)D_qP_n(y)dy.$$

Applying Lemma 1.2, (1.13), and (1.14) we see that

$$\begin{aligned} q\zeta_k c_{n,k} &= -\int_0^\infty P_n(y)\left[(D_{q^{-1}}P_k(y))w(y) + P_k(y/q)D_{q^{-1}}w(y)\right]dy \\ &= \int_0^\infty P_n(y)P_k(y/q)\left[-\frac{D_{q^{-1}}w(y)}{w(y)}\right]w(y)dy \end{aligned}$$

where the orthogonality property was used in the last step. The definition of u (1.2) yields

$$\begin{aligned} q\zeta_k c_{n,k} &= \int_0^\infty P_n(y)P_k(y/q)(u(y))w(y)dy \\ &= -\int_0^\infty P_n(y)P_k(y/q)(u(qx)-u(y))w(y)dy, \end{aligned}$$

where we again used the orthogonality property in the last step. Therefore

$$D_qP_n(x) = -\frac{1}{\zeta_{n-1}}\int_0^\infty P_n(y)\frac{u(qx)-u(y)}{qx-y}$$
$$[P_n(x)P_{n-1}(y/q) - P_n(y/q)P_{n-1}(x)]\, w(y)dy$$

and the theorem now follows from the Christoffel–Darboux identity (2.2) and (2.1). ☐

Proof of (1.9). It is clear that

$$\begin{aligned} &B_{n+1}(x) + B_n(x) \\ &= \int_0^\infty \frac{u(qx)-u(x)}{\zeta_n(qx-y)}\left[P_{n+1}(y)P_n(y/q) + \beta_nP_n(y)P_{n-1}(y/q)\right] w(y)dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}\,(y/q-\alpha_n)P_n(y)\,P_n(y/q)w(y)dy \\ I_2 &:= \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}\left[P_{n+1}(y)P_n(y/q) - P_n(y)P_{n+1}(y/q)\right] w(y)dy, \end{aligned}$$

After $\beta_nP_{n-1}(y/q)$ is replaced by $(y/q-\alpha_n)P_n(y/q) - P_{n+1}(y/q)$. It is easy to see that I_1 is given by

$$\begin{aligned} I_1 &= (x-\alpha_n)A_n(x) - \frac{1}{\zeta_nq}\int_0^\infty (u(qx)-u(y))P_n(y)P_n(y/q)w(y)dy \\ &= (x-\alpha_n)A_n(x) - q^{-n-1}u(qx), \end{aligned}$$

where (1.15) and the fact that

$$P_j(y/q) = q^{-j}P_j(y) + \text{lower degree terms} \tag{2.3}$$

were used. To evaluate I_2 first note that (2.3) implies

$$\int_0^\infty P_j(y)P_j(y/q)w(y)dy = \zeta_jq^{-j}. \tag{2.4}$$

Next we apply the Christoffel–Darboux formula to

$$[P_{n+1}(y)P_n(y/q) - P_n(y)P_{n+1}(y/q)],$$

and replace $y - y/q$ by $(y - qx + qx)(1 - 1/q)$. Therefore we see that

$$I_2 = x(q-1)\sum_{j=0}^{n} A_j(x) + \frac{1-q}{q}\int_0^{\infty}[u(qx) - u(y)]\sum_{j=0}^{n}\frac{P_j(y)P_j(y/q)}{\zeta_j}w(y)dy$$

$$= x(q-1)\sum_{j=0}^{n} A_j(x) + \frac{1-q}{q}u(qx)\sum_{j=0}^{n} q^{-j} + \frac{1-q}{q}\int_0^{\infty}\sum_{j=0}^{n}\frac{P_j(y)P_j(y/q)}{\zeta_j}D_{q^{-1}}w(y)\,dy.$$

Thus

$$I_2 = x(q-1)\sum_{j=0}^{n} A_j(x) + \frac{1-q}{q}u(qx)\frac{1-q^{-n-1}}{1-q^{-1}}.$$

Simplifying $I_1 + I_2$ we establish (1.9). $\square$

Proof of (1.10). From the definition of $B_n(x)$ we see that

$$
\begin{aligned}
&(x-\alpha_n)B_{n+1}(x)-(qx-\alpha_n)B_n(x)\\
&=\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}\\
&\times\left[\left(\frac{x-\alpha_n}{\zeta_n}\right)P_{n+1}(y)P_n(y/q)-\left(\frac{qx-\alpha_n}{\zeta_{n-1}}\right)P_n(y)P_{n-1}(y/q)\right]dy\\
&=\int_0^\infty w(y)\left[u(qx)-u(y)\right]\left[\frac{1}{\zeta_n}\left(\frac{1}{q}+\frac{y/q-\alpha_n}{qx-y}\right)P_{n+1}(y)P_n(y/q)\right.\\
&\left.-\frac{1}{\zeta_{n-1}}\left(1+\frac{y-\alpha_n}{qx-y}\right)P_n(y)P_{n-1}(y/q)\right]dy\\
&=-\frac{1}{\zeta_n q}\int_0^\infty w(y)u(y)P_{n+1}(y)P_n(y/q)dy\\
&+\frac{1}{\zeta_n}\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}(y/q-\alpha_n)P_n(y/q)P_{n+1}(y)dy\\
&+\frac{1}{\zeta_{n-1}}\int_0^\infty w(y)u(y)P_n(y)P_{n-1}(y/q)dy\\
&-\frac{1}{\zeta_{n-1}}\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}P_n(y)P_{n-1}(y/q)(y-\alpha_n)dy\\
&=-\frac{1}{\zeta_n q}\int_0^\infty w(y)u(y)P_{n+1}(y)P_n(y/q)dy\\
&+\frac{1}{\zeta_n}\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}\left[P_{n+1}(y/q)+\beta_nP_{n-1}(y/q)\right]P_{n+1}(y)dy\\
&+\frac{1}{\zeta_{n-1}}\int_0^\infty w(y)u(y)P_n(y)P_{n-1}(y/q)dy\\
&-\frac{1}{\zeta_{n-1}}\int_0^\infty w(y)\frac{u(qx)-u(y)}{qx-y}\left(P_{n+1}(y)+\beta_nP_{n-1}(y)\right)P_{n-1}(y/q)dy.
\end{aligned}
$$

The result follows after some simplifications using (1.16). ☐

3 Stieltjest-Wigert polynomials

This is example of an indeterminate moment problem associated with the log-normal density. We take the weight to be

$$w(x)=c\exp[(\ln x)^2/(2\ln q)],\quad 0\le x<\infty,\quad 0<q<1,$$

where c is a positive constant which will not appear in subsequent computations.

An easy calculation shows that

$$\begin{aligned} u(x) &= \frac{q}{1-q}\left(\frac{1}{x}-\frac{\sqrt{q}}{x^2}\right), \\ A_n(x) &= \frac{R_n}{x^2}, \\ \text{where } R_n &:= \frac{1}{\zeta_n(1-q)\sqrt{q}}\int_0^\infty P_n(y)P_n(y/q)w(y)\frac{dy}{y}, \\ B_n(x) &= \frac{r_n}{x^2}-\frac{1-q^n}{1-q}\frac{1}{x} \\ \text{where } r_n &:= \frac{1}{\zeta_{n-1}\sqrt{q}(1-q)}\int_0^\infty P_n(y)P_{n-1}(y/q)w(y)\frac{dy}{y}. \end{aligned}$$

From the supplementary conditions, (1.9) and (1.10)

$$\begin{aligned} &(3.1) & \frac{q^{n+1}+q^n-2}{1-q} &= R_n+(q-1)S_n-\frac{1}{1-q} \\ &(3.2) & r_{n+1}+r_n &= -\alpha_n R_n+\frac{1}{\sqrt{q}(1-q)} \\ &(3.3) & 0 &= \alpha_n\frac{1-q^{n+1}}{1-q}-\alpha_n\frac{1-q^n}{1-q}+r_{n+1}-qr_n \\ &(3.4) & \alpha_n(r_n-r_{n+1}) &= \beta_{n+1}R_{n+1}-\beta_n R_{n-1}, \end{aligned}$$

where $S_n := \sum_{j=0}^n R_j$, and

$$R_0 = \frac{1}{\sqrt{q}(1-q)}\frac{\int_0^\infty w(y)/y\,dy}{\int_0^\infty w(y)dy} = \frac{1}{1-q}. \tag{3.5}$$

A difference equation satisfied by R_n is found by subtracting (3.1) at "$n-1$" from the same at "n";

$$qR_n - R_{n-1} = -(1+q)q^n, \tag{3.6}$$

and since the "integrating factor" is q^{-n}, the unique solution is

$$R_n = \frac{q^n}{1-q}. \tag{3.7}$$

Note that (3.3) simplifies to

$$-\alpha_n q^n = r_{n+1}-r_n. \tag{3.8}$$

Multiplying (3.2) by $1-q$, together with $R_n(1-q)=q^n$ and (3.8) one finds

$$r_n - q r_{n+1} = \frac{1}{\sqrt{q}}, \tag{3.9}$$

and since the "integrating factor" for this is q^n, the unique solution subject to $r_0 = 0$, is

$$r_n = \frac{1-q^{-n}}{(1-q)\sqrt{q}}, \tag{3.10}$$

which with (3.8) immediately gives,

$$\alpha_n = \frac{q^{-n}}{\sqrt{q}}\left(q^{-n-1}+q^{-n}-1\right). \tag{3.11}$$

Multiply (3.4) by R_n and replace $\alpha_n R_n$ with (3.2), we find the resulting first order difference equation

$$r_{n+1}^2 - \frac{r_{n+1}}{\sqrt{q}(1-q)} - \left(r_n^2 - \frac{r_n}{\sqrt{q}(1-q)}\right) = \beta_{n+1}R_{n+1}R_n - \beta_n R_n R_{n-1},$$

where the solution with the initial conditions $r_0 = \beta_0 = 0$ is

$$r_n^2 - \frac{r_n}{\sqrt{q}(1-q)} = \beta_n R_n R_{n-1}. \tag{3.12}$$

This expresses β_n in terms of the subsidiary quantities r_n and R_n,

$$\begin{aligned}\beta_n &= \frac{r_n}{R_n R_{n-1}}\left(r_n - \frac{1}{\sqrt{q}(1-q)}\right)\\ &= q^{-4n} - q^{-3n}. \end{aligned}\tag{3.13}$$

In the next section we take a route for the computations of the recurrence coefficients which does not involve the determination of the analogous r_n and R_n.

4 q-Laguerre polynomials

This is also associated with an indeterminate moment problem at a level "higher" than the Stieltjest-Wigert polynomials. W"e take the weight to be,

$$w(x) = \frac{x^\alpha}{(-x;q)_\infty}, \quad 0 \le x < \infty, \quad \alpha > -1, \quad 0 < q < 1, \tag{4.1}$$

where

$$(z;q)_\infty := (1-z)(1-qz)...$$

This weight leads to

$$\begin{aligned}
u(x) &= \frac{q}{1-q}\left(\frac{1-q^{-\alpha}}{x}+\frac{q^{-\alpha}}{x+q}\right)\\
\frac{u(qx)-u(y)}{qx-y} &= \frac{1}{1-q}\left(\frac{q^{-\alpha}-1}{xy}-\frac{q^{-\alpha}}{(x+1)(y+q)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_n(x) &= \frac{q^{-\alpha}-1}{\zeta_n(1-q)x}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y}dy\\
&- \frac{q^{-\alpha}}{\zeta_n(1-q)(x+1)}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y+q}dy\\
&=: \frac{R_n}{x}-\frac{q^n}{(1-q)(x+1)} \qquad (4.2)\\
B_n(x) &= \frac{q^{-\alpha}-1}{(1-q)\zeta_{n-1}x}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}dy\\
&- \frac{q^{-\alpha}}{\zeta_{n-1}(1-q)(x+1)}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y+q}dy\\
&=: \frac{r_n}{x}-\frac{q^{n-1}\mathsf{p}_1(n)}{x+1}. \qquad (4.3)
\end{aligned}$$

Before we evaluate the second integrals in $A_n(x)$ and $B_n(x)$, here is a computation of R_0. By definition,

$$\begin{aligned}
R_0 &:= \frac{q^{-\alpha}-1}{1-q}\,\frac{\int_0^\infty w(y)dy/y}{\int_0^\infty w dy}\\
&= \frac{q^{-\alpha}-1}{1-q}\,\frac{I(\alpha)}{I(\alpha+1)} = \frac{1}{1-q},
\end{aligned}$$

since

$$I(\alpha) := \int_0^\infty \frac{y^{\alpha-1}}{(-y;q)_\infty}dy = \frac{(q^{1-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi\alpha}.$$

Also note the identity,

$$\frac{1}{(-x;q)_\infty(x+q)} = \frac{1}{q(1+x/q)(-x/q;q)_\infty} = \frac{1}{q(-x/q;q)_\infty}.$$

Hence,

$$\begin{aligned}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y+q}dy &= \int_0^\infty P_n(y)P_n(y/q)\frac{y^\alpha}{q(-y/q;q)_\infty}dy \\ &= q^\alpha\int_0^\infty P_n(qy)P_n(y)w(y)dy = q^{n+\alpha}\zeta_n,\end{aligned}$$

and the result for $A_n(x)$ follows. Similarly,

$$\begin{aligned}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y+q}dy &= \int_0^\infty P_n(y)P_{n-1}(y/q)\frac{y^\alpha}{q(-y/q;q)_\infty}dy \\ &= q^\alpha\int_0^\infty P_n(qy)P_{n-1}(y)w(y)dy.\end{aligned}$$

To complete the evaluation of the above integral, we note the identity,

$$\begin{aligned}P_n(qy) &= P_n(qy)+q^nP_n(y)-q^nP_n(y) \\ &= q^nP_n(y)+q^ny^n+q^{n-1}\mathsf{p}_1(n)y^{n-1}+...-q^n\left(y^n+\mathsf{p}_1(n)y^{n-1}+...\right) \\ &= q^nP_n(y)+\mathsf{p}_1(n)(q^{n-1}-q^n)y^{n-1}+... \\ &= q^nP_n(y)+\mathsf{p}_1(n)(q^{n-1}-q^n)P_{n-1}(y)+...\end{aligned}$$

Finally,

$$\begin{aligned}& q^\alpha\int_0^\infty P_n(qy)P_{n-1}(y)w(y)dy \\ = {}& q^\alpha\int_0^\infty\{q^nP_n(y)+\mathsf{p}_1(n)(q^{n-1}-q^n)P_{n-1}(y)+...\}P_{n-1}(y)w(y)dy \\ = {}& \left(\frac{1}{q}-1\right)\mathsf{p}_1(n)q^{n+\alpha}\zeta_{n-1},\end{aligned}$$

and the result for $B_n(x)$ follows.

It turns out that for the $q-Laguerre$ weight, the supplementary conditions produce 6 difference equations in contrast with the 4 in the previous example.

Now, by equating the residues for the simple poles at $x=0$ and $x=-1$ in (1.9), we find,

$$r_{n+1}+r_n = -\alpha_nR_n-\frac{1-q^{-\alpha}}{1-q} \tag{4.4}$$

$$\mathsf{p}_1(n+1)q^n+\mathsf{p}_1(n)q^{n-1} = -\frac{1+\alpha_n}{1-q}q^n+\frac{1-q^{n+1}}{1-q}+\frac{q^{-\alpha}}{1-q}, \tag{4.5}$$

respectively. We note here another identity involving R_n only, by equating the constant terms of (1.9) at $x = \infty$,

$$R_n + \frac{q^n}{1-q} + (q-1)S_n + \frac{1-q^{n+1}}{1-q} = 0, \tag{4.6}$$

where $S_n = \sum_{j=0}^{n} R_j$. A similar consideration on (1.10) shows that

$$\alpha_n(r_n - r_{n+1}) = \beta_{n+1}R_{n+1} - \beta_n R_{n-1} \tag{4.7}$$

$$-(1+\alpha_n)q^n \mathsf{p}_1(n+1) + (q+\alpha_n)q^{n-1}\mathsf{p}_1(n) = \frac{\beta_{n+1}q^{n+1} - \beta_n q^{n-1}}{1-q} \tag{4.8}$$

$$r_{n+1} - qr_n = -q^n\alpha_n - 1. \tag{4.9}$$

We use the fact that $\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1)$ to rewrite (4.5) as a first order difference equation,

$$\mathsf{p}_1(n+1)q^{n+1} - \mathsf{p}_1(n)q^{n-1} = q^n + q^{n+1} - (1+q^{-\alpha})$$

which has an "integrating factor" q^{n-1} by inspection. Hence the above equation becomes,

$$\mathsf{p}_1(n+1)q^{2n} - \mathsf{p}_1(n)q^{2n-2} = (1+q)q^{2n-1} - (1+q^{-\alpha})q^{n-1}, \tag{4.10}$$

and we find via a telescopic sum and the initial condition $\mathsf{p}_1(0) = 0$,

$$\mathsf{p}_1(n+1)q^{2n} = \frac{1-q^{2n+2}}{q(1-q)} - (1+q^{-\alpha})\frac{1-q^{n+1}}{q(1-q)}.$$

Therefore,

$$(1-q)\mathsf{p}_1(n) = -q + (1+q^{-\alpha})q^{-n+1} - q^{-2n-\alpha+1}, \tag{4.11}$$

and equation (1.5) gives

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1) = q^{-2n-1-\alpha}(1+q-q^{n+1}-q^{n+\alpha+1}). \tag{4.12}$$

At this stage R_n can be found a difference equation obtained by subtracting (4.6) at "n" from the same at "$n+1$",

$$qR_{n+1} - R_n = q^{n+1} - q^n, \tag{4.13}$$

with the initial condition $R_0 = 1/(1-q)$. Having now determined α_n in (4.12), r_n can be found from (4.9) with the initial condition $r_0 = 0$. We proceed to the determination of β_n. Multiply (4.8) by the "integrating factor" q^n and by $1-q$,

$$\begin{aligned} &- (1+\alpha_n)q^{2n}(1-q)\mathsf{p}_1(n+1) + (q+\alpha_n)q^{2n-1}(1-q)\mathsf{p}_1(n) \\ &= \beta_{n+1}q^{2n+1} - \beta_n q^{2n-1}. \end{aligned} \tag{4.14}$$

The l.h.s. of the above is simplified to,

$$(1-q)q^{2n}\left(\mathsf{p}_1(n)-\mathsf{p}_1(n+1)\right)-\alpha_n(1-q)\left(\mathsf{p}_1(n+1)q^{2n}-\mathsf{p}_1(n)q^{2n-1}\right)$$
$$=(1-q)q^{2n}\alpha_n\left[1-(\mathsf{p}_1(n+1)-\mathsf{p}_1(n)/q)\right].$$

With (4.12), the term $\mathsf{p}_1(n+1)-\mathsf{p}_1(n)/q$ simplifies to

$$1-q^{-2n-\alpha-1},$$

and consequently (4.14) becomes to the first order difference equation, with the initial condition $\beta_0=0$,

$$\beta_{n+1}q^{2n+1}-\beta_n q^{2n-1}=(1-q)q^{-1-\alpha}\alpha_n. \tag{4.15}$$

Taking a telescopic sum, and noting that $\sum_{j=0}^{n-1}\alpha_j=-\mathsf{p}_1(n)$,

$$\beta_n q^{2n-1} \quad = \quad (1-q)q^{-1-\alpha}\sum_{j=0}^{n-1}\alpha_n=-(1-q)q^{-1-\alpha}\mathsf{p}_1(n). \tag{4.16}$$

Finally,

$$\begin{aligned}\beta_n &= -q^{-2n-\alpha}(1-q)\mathsf{p}_1(n)\\ &= -q^{-2n-\alpha}(-q+(1+q^{-\alpha})q^{1-n}-q^{-2n-\alpha+1})\\ &= q^{-4n-2\alpha+1}(1-q^n)(1-q^{n+\alpha}). \qquad (4.17)\end{aligned}$$

It is interesting to note that in the computations of α_n and β_n, not all the 6 equations are required. We have used only (4.5) and (4.8). However, for an explicit expression of the $q-$Ladder operators and therefore the determination of r_n and R_n we need the equations (4.6), (4.9) and α_n in (4.12).

We end with the remark that in the case of the classical Laguerre polynomials, $\alpha_n=2n+1+\alpha$, and $\beta_n=\sum_{j=0}^{n-1}\alpha_j=n(n+\alpha)$, and this is analogous to (4.16), however, with appropriate modifications in the $q-$case.

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