Component sizes of the random graph outside the scaling window

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Abstract

We provide simple proofs describing the behavior of the largest component of the Erdős-Rényi random graph $G(n,p)$ outside of the scaling window, $p = \frac{1+\epsilon(n)}{n}$ where $\epsilon(n) \to 0$ but $\epsilon(n)n^{1/3} \to \infty$.

1 Introduction

Consider the random graph $G(n,p)$ obtained from the complete graph on n vertices by retaining each edge with probability p and deleting each edge with probability $1-p$. We denote by $\mathcal{C}_j$ the j-th largest component. Let $\epsilon(n)$ be a non-negative sequence such that $\epsilon(n) \to 0$ and $\epsilon(n)n^{1/3} \to \infty$. The following theorems, proved by Bollobás [4] and Łuczak [8] using different methods, describe the behavior of the largest components when p is outside the "scaling-window".

Theorem 1 **[Subcritical phase]** *If* $p(n) = \frac{1-\epsilon(n)}{n}$ *then for any* $\eta > 0$ *and integer* $\ell > 0$ *we have*

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon(n)^{-2}\log(n\epsilon(n)^3)} - 1\Big| > \eta\Big) \to 0\,,$$

as $n \to \infty$.

Theorem 2 **[Supercritical phase]** *If* $p(n) = \frac{1+\epsilon(n)}{n}$ *then for any* $\eta > 0$ *we have*

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_1|}{2n\epsilon(n)} - 1\Big| > \eta\Big) \to 0\,,$$

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and for any integer $\ell > 1$ *we have*

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon^{-2}(n)\log(n\epsilon^3)} - 1\Big| > \eta\Big) \to 0\,,$$

as $n \to \infty$.

The proofs of these theorems in [4] and [8] are quite involved and use the detailed asymptotics from [14], [4] and [3] for the number of graphs on k vertices with $k+\ell$ edges. The proofs we present here are simple and require no hard theorems. The main advantage, however, of these proofs is their robustness. In a companion paper [12] we use similar methods to analyze critical percolation on a random regular graphs. In this case, the enumerative methods employed in [4] and [8] are not available.

The phase transition in the Erdős-Rényi random graphs $G(n,p)$ happens when $p = \frac{c}{n}$. Namely, with high probability, if $c > 1$ then $|\mathcal{C}_1|$ is linear in n, and if $c < 1$ then $|\mathcal{C}_1|$ is logarithmic in n. When $c \sim 1$ the situation is more delicate. In [9], Łuczak, Pittel and Wierman prove that for $p = \frac{1+\lambda n^{-1/3}}{n}$, the law of $n^{-2/3}|\mathcal{C}_1|$ converges to a positive non-constant distribution which in [1] is identified as the longest excursion length of some Brownian motion with variable drift. See [11] for a recent account of the case $p = \frac{1+\lambda n^{-1/3}}{n}$ with simple proofs.

Thus, $|\mathcal{C}_1|$ is not concentrated and is roughly of size $n^{2/3}$ if $p = \frac{1+\lambda n^{-1/3}}{n}$. However, if $\epsilon(n)$ a sequence such that $n^{1/3}\epsilon(n) \to \infty$ and $p = \frac{1+\epsilon(n)}{n}$ then as stated in Theorems 1 and 2, the size $|\mathcal{C}_1|$ of the largest component in $G(n,p)$ is concentrated. In summary, $G(n,p)$ has a scaling window of length $n^{-1/3}$ in which the percolation is "critical" in the sense that $|\mathcal{C}_1|$ is not concentrated.

2 The exploration process

We recall an exploration process, due to Karp and Martin-Löf (see [7] and [10]), in which vertices will be either *active, explored* or *neutral.* After the completion of step $t \in \{0,1,\ldots,n\}$ we will have precisely t explored vertices and the number of the active and neutral vertices is denoted by A_t and N_t respectively.

Fix an ordering of the vertices $\{v_1,\ldots,v_n\}$. In step $t = 0$ of the process, we declare vertex v_1 *active* and all other vertices *neutral.* Thus $A_0 = 1$ and $N_0 = n-1$. In step $t \in \{1,\ldots,n\}$, if $A_{t-1} > 0$ let w_t be the first active vertex; if $A_{t-1} = 0$, let w_t be the first neutral vertex. Denote by η_t the number of neutral neighbors of w_t in $G(n,p)$, and change the status of these vertices to *active*. Then, set w_t itself *explored*.

Denote by $\mathcal{F}_t$ the σ-algebra generated by $\{\eta_1,\ldots,\eta_t\}$. Observe that given $\mathcal{F}_{t-1}$ the random variable η_t is distributed as $\mathrm{Bin}(N_{t-1}-\mathbf{1}_{\{A_{t-1}=0\}},p)$ and we have the recursions

$$N_t = N_{t-1} - \eta_t - \mathbf{1}_{\{A_{t-1}=0\}}\,, \qquad t \le n\,, \tag{1}$$

and

$$A_t = \begin{cases} A_{t-1}+\eta_t-1, & A_{t-1}>0 \\ \eta_t, & A_{t-1}=0\,, \qquad t\le n\,. \end{cases} \tag{2}$$

As every vertex is either neutral, active or explored,

$$N_t = n - t - A_t\,, \qquad t \le n\,. \tag{3}$$

At each time $j \le n$ in which $A_j = 0$, we have finished exploring a connected component. Hence the random variable Z_t defined by

$$Z_t = \sum_{j=1}^{t-1} \mathbf{1}_{\{A_j=0\}}\,,$$

counts the number of components completely explored by the process before time t. Define the process $\{Y_t\}$ by $Y_0 = 1$ and

$$Y_t = Y_{t-1} + \eta_t - 1\,.$$

By (2) we have that $Y_t = A_t - Z_t$, i.e. Y_t counts the number of active vertices at step t minus the number of components completely explored before step t.

At each step we marked as explored precisely one vertex. Hence, the component of v_1 has size $\min\{t \ge 1 : A_t = 0\}$. Moreover, let $t_1 < t_2 \ldots$ be the times at which $A_{t_j} = 0$; then $(t_1, t_2-t_1, t_3-t_2, \ldots)$ are the sizes of the components. Observe that $Z_t = Z_{t_j}+1$ for all $t \in \{t_j+1,\ldots,t_{j+1}\}$. Thus $Y_{t_{j+1}} = Y_{t_j} - 1$ and if $t \in \{t_j+1,\ldots,t_{j+1}-1\}$ then $A_t > 0$, and thus $Y_{t_{j+1}} < Y_t$. By induction we conclude that $A_t = 0$ if and only if $Y_t < Y_s$ for all $s < t$, i.e. $A_t = 0$ if and only if $\{Y_t\}$ has hit a new record minimum at time t. By induction we also observe that $Y_{t_j} = -(j-1)$ and that for $t \in \{t_j+1,\ldots t_{j+1}\}$ we have $Z_t = j$. Also, by our previous discussion for $t \in \{t_j+1,\ldots t_{j+1}\}$ we have $\min_{s\le t-1} Y_t = Y_{t_j} = -(j-1)$, hence by induction we deduce that $Z_t = -\min_{s\le t-1} Y_t + 1$. Consequently,

$$A_t = Y_t - \min_{s \le t-1} Y_s + 1\,. \tag{4}$$

Lemma 3 *For all $p \leq \frac{2}{n}$ there exists a constant $c > 0$ such that for any integer $t > 0$,*

$$\mathbf{P}\Big(N_t \leq n - 5t\Big) \leq e^{-ct} .$$

Proof. Let $\{\alpha_i\}_{i=1}^t$ be a sequence of i.i.d. random variables distributed as $\mathrm{Bin}(n,p)$. It is clear that we can couple η_i and α_i so $\eta_i \leq \alpha_i$ for all i, and thus by (1)

$$N_t \geq n - 1 - t - \sum_{i=1}^t \alpha_i . \tag{5}$$

The sum $\sum_{i=1}^t \alpha_i$ is distributed as $\mathrm{Bin}(nt,p)$ and $p \leq \frac{2}{n}$ so by Large Deviations (see [2] section A.14) we get that for some fixed $c > 0$

$$\mathbf{P}\Big(\sum_{i=1}^t \alpha_i \geq 3t\Big) \leq e^{-ct} ,$$

which together with (5) concludes the proof. □

3 The subcritical phase

Before beginning the proof of Theorem 1 we require some facts about processes with i.i.d. increments. Fix some small $\epsilon > 0$ and let $p = \frac{1-\epsilon}{m}$ for some integer $m > 1$. Let $\{\beta_j\}$ be a sequence of random variables distributed as $\mathrm{Bin}(m,p)$. Let $\{W_t\}_{t\geq 0}$ be a process defined by

$$W_0 = 1, \qquad W_t = W_{t-1} + \beta_t - 1 .$$

Let τ be the hitting time of 0,

$$\tau = \min_t \{W_t = 0\} .$$

By Wald's lemma we have that $\mathbf{E}\,\tau = \epsilon^{-1}$. Further information on the tail distribution of τ is given by the following lemma.

Lemma 4 *There exists constant $C_1, C_2, c_1, c_2 > 0$ such that for all $T > \epsilon^{-2}$ we have*

$$\mathbf{P}(\tau \geq T) \leq C_1\Big(\epsilon^{-2}T^{-3/2}e^{-\frac{(\epsilon^2 - c_1\epsilon^3)T}{2}}\Big) ,$$

and

$$\mathbf{P}(\tau \geq T) \geq c_1\Big(\epsilon^{-2}T^{-3/2}e^{-\frac{(\epsilon^2 + c_2\epsilon^3)T}{2}}\Big) .$$

Furthermore,

$$\mathbf{E}\,\tau^2 = O(\epsilon^{-3}) .$$

We will use the following proposition due to Spitzer (see [13]).

Proposition 5 *Let $a_0, \ldots, a_{k-1} \in \mathbb{Z}$ satisfy $\sum_{i=0}^{k-1} a_i = -1$. Then there is precisely one $j \in \{0, \ldots, k-1\}$ such that for all $r \in \{0, \ldots, k-2\}$*

$$\sum_{i=0}^{r} a_{(j+i) \bmod k} \geq 0 \,.$$

Proof of Lemma 4. By Proposition 5, $\mathbf{P}(\tau = t) = \frac{1}{t}\mathbf{P}(W_t = 0)$. As $\sum_{j=1}^{t} \beta_j$ is distributed as a $\mathrm{Bin}(mt, p)$ random variable we have

$$\mathbf{P}(W_t = 0) = \binom{mt}{t-1} p^{t-1}(1-p)^{m-(t-1)} \,.$$

Replacing $t-1$ with t in the above formula only changes it by a multiplicative constant which is always between $1/2$ and 2. A straightforward computation using Stirling's approximation gives

$$\mathbf{P}(W_t = 0) = \Theta\Big\{ t^{-1/2}(1-\epsilon)^t \Big(1 + \frac{1}{m-1}\Big)^{tm} \Big(1 - \frac{1-\epsilon}{m}\Big)^{t(m-1)} \Big\} \,. \qquad (6)$$

Denote $q = (1-\epsilon)\Big(1 + \frac{1}{m-1}\Big)^m \Big(1 - \frac{1-\epsilon}{m}\Big)^{m-1}$, then

$$\mathbf{P}(\tau \geq T) = \sum_{t \geq T} \mathbf{P}(\tau = t) = \sum_{t \geq T} \frac{1}{t}\mathbf{P}(W_t = 0) = \Theta\Big(\sum_{t \geq T} t^{-3/2} q^t \Big) \,.$$

This sum can be bounded above by

$$T^{-3/2} \sum_{t \geq T} q^t = T^{-3/2} \frac{q^T}{1-q} \,,$$

and below by

$$\sum_{t=T}^{2T} t^{-3/2} q^t \geq (2T)^{-3/2} \frac{q^T(1-q^T)}{1-q} \,.$$

Observe that as $m \to \infty$ we have that q tends to $(1-\epsilon)e^\epsilon$. By expanding e^ϵ we find that

$$q = (1-\epsilon)(1 + \epsilon + \frac{\epsilon^2}{2}) + \Theta(\epsilon^3) = 1 - \frac{\epsilon^2}{2} + \Theta(\epsilon^3) \,.$$

Using this and the previous bounds on $\mathbf{P}(\tau \geq T)$ we get the first assertion of the Lemma.

The second assertion follows from the following computation. By (6) we have that for some constant $C > 0$

$$\mathbf{E}\,\tau^2 = \sum_{t\geq 1} t^2 \mathbf{P}(\tau = t) = \sum_{t\geq 1} t\mathbf{P}(W_t = 0) \leq C \sum_{t\geq 1} \sqrt{t} q^t \,.$$

Thus, by direct computation (or by [6], section XIII.5, Theorem 5)

$$\mathbf{E}\,\tau^2 \leq O\Big(\frac{1}{1-q}\Big)^{3/2} = O(\epsilon^{-3})\,.$$

□

Proof of Theorem 1. We begin with an upper bound. Recall that component sizes are $t_{j+1} - t_j$ for some $j > 0$ where t_j are record minima of the process $\{Y_t\}$. For a vertex v denote by $C(v)$ the connected component of $G(n, p)$ which contains v. We first bound $\mathbf{P}(|C(v_1)| > T)$ where

$$T = 2(1+\eta)\epsilon^{-2} \log(n\epsilon^3)\,.$$

Recall that $|C(v_1)| = \min_t\{Y_t = 0\}$. Couple $\{Y_t\}$ with a process $\{W_t\}$ as in Lemma 4, which has increments distributed as $\mathrm{Bin}(n, p) - 1$ such that $Y_t \leq W_t$ for all t. Define τ as in Lemma 4. As $p = \frac{1-\epsilon}{n}$ and $T > \epsilon^{-2}$, by Lemma 4 we have

$$\mathbf{P}(\tau > T) \leq C\epsilon(n\epsilon^3)^{-(1+\eta)} \log(n\epsilon^3)^{-3/2}\,,$$

for some fixed $C > 0$. Our coupling implies that $\mathbf{P}(|C(v_1)| > T) \leq \mathbf{P}(\tau > T)$. Denote by X the number of vertices v such that $|C(v)| > T$. If $|\mathcal{C}_1| > T$ then $X > T$. Also, for any two vertices v and u by symmetry we have that $|C(v)|$ and $|C(u)|$ are identically distributed. We conclude that

$$\begin{aligned}\mathbf{P}(|\mathcal{C}_1| > T) &\leq \mathbf{P}(X > T) \leq \frac{\mathbf{E}\,X}{T} = \frac{n\mathbf{P}(|C(v_1)| > T)}{T} \\ &\leq \frac{C_1 n\epsilon(n\epsilon^3)^{-(1+\eta)(1-C_2\epsilon)} \log^{-3/2}(n\epsilon^3)}{2(1+\eta)\epsilon^{-2}\log(n\epsilon^3)} \leq (n\epsilon^3)^{-\eta(1-C_2\epsilon)+C_2\epsilon} \to 0\,.\end{aligned}$$

We now turn to prove a lower bound. Write

$$T = 2(1-\eta)\epsilon^{-2} \log(n\epsilon^3)\,,$$

and define the stopping time

$$\gamma = \min\{t : N_t \le n - \frac{\eta\epsilon n}{8}\}\,.$$

Recall that $\{t_j\}$ are times in which $A_{t_j} = 0$ and also Y_{t_j} is a record minimum for $\{Y_t\}$. For each integer j let $\{W_t^{(j)}\}$ be a process with increments distributed as $\mathrm{Bin}(n - \frac{\eta\epsilon n}{8}, p)$ where the starting point is $W_0^{(j)} = Y_{t_j} = -(j-1)$. Note that if $t_{j+1} < \gamma$ then we can couple $\{Y_t\}$ and $\{W_t^{(j)}\}$ such that $Y_{t_j+t} \ge W_t$ for all $t \in [t_j, t_{j+1}]$. Define the stopping times $\{\tau_j\}$ by

$$\tau_j = \min\{t : W_t^{(j)} = -j\}\,.$$

Take

$$N = \left\lceil \epsilon^{-1}(n\epsilon^3)^{(1-\frac{\eta}{8})} \right\rceil .$$

We will prove that with high probability $t_N < \gamma$ and that there exists $k_1 < k_2 < \ldots < k_\ell < N$ such that $\tau_{k_i} > T$. Note that these two events imply that $|\mathcal{C}_\ell| > T$. Indeed, by Lemma 3 we have

$$\mathbf{P}\Big(\gamma \le \frac{\eta\epsilon n}{40}\Big) \le e^{-c\epsilon n}\,. \tag{7}$$

By bounding the increments of $\{Y_t\}$ above by variables distributed as $\mathrm{Bin}(n,p)-1$ we learn by Wald's Lemma (see [5]) that $\mathbf{E}\,[t_{j+1} - t_j] \le \epsilon^{-1}$, hence $\mathbf{E}\,t_N \le \epsilon^{-2}(n\epsilon^3)^{(1-\frac{\eta}{8})}$. We conclude that

$$\mathbf{P}(t_N > \frac{\eta\epsilon n}{40}) \le \frac{40\epsilon^{-2}(n\epsilon^3)^{(1-\frac{\eta}{8})}}{\eta\epsilon n} = \frac{40}{\eta}(n\epsilon^3)^{-\frac{\eta}{8}}\,, \tag{8}$$

which goes to 0 as $\epsilon n^{-1/3}$ tends to ∞. In Lemma 4 take $m = n - \frac{\eta\epsilon n}{8}$ and note that $p = \frac{(1-\epsilon)(1-\frac{\eta\epsilon}{8})}{m} \ge \frac{1-(1+\frac{\eta}{8})\epsilon}{m}$, and so Lemma 4 gives that for any j

$$\mathbf{P}(\tau_j > T) \ge c_1\epsilon(n\epsilon^3)^{-(1+\frac{\eta}{8})^2(1-\eta)(1+c_2\epsilon)}\log^{-3/2}(\epsilon^3 n) \ge \epsilon(n\epsilon^3)^{-(1-\frac{\eta}{4})}\,.$$

Let X be the number of $j \le N$ such that $\tau_j > T$. Then we have

$$\mathbf{E}\,X \ge N\epsilon(n\epsilon^3)^{-(1-\frac{\eta}{4})} \ge C(n\epsilon^3)^{\frac{\eta}{8}} \to \infty\,,$$

hence by Large Deviations (see [2], section A.14) for any fixed integer $\ell > 0$ we have

$$\mathbf{P}\Big(X < \ell\Big) \le e^{-c(n\epsilon^3)^{\frac{\eta}{8}}}\,,$$

for some fixed $c > 0$. By our previous discussion, this together with (7) and (8) gives

$$\mathbf{P}(|\mathcal{C}_\ell| < T) \leq O\Big(\frac{(n\epsilon^3)^{-\frac{\eta}{8}}}{\eta}\Big)\,.$$

□

4 The supercritical phase

In this section we denote $\xi_t = \eta_t - 1$. We first prove some Lemmas.

Lemma 6 *If $p = \frac{1+\epsilon}{n}$ then for all $t < 3\epsilon(n)n$*

$$\mathbf{E}\,A_t = O(\epsilon t + \sqrt{t})\,, \tag{9}$$

and

$$\mathbf{E}\,Z_t = O(\epsilon t + \sqrt{t})\,. \tag{10}$$

Proof. Write $T = 3\epsilon n$. We will use (4). First observe that as η_t can always be bounded above by a $\text{Bin}(n,p)$ random variable we can bound $\mathbf{E}\,\xi_t \leq \epsilon$ for all t and hence $\mathbf{E}\,Y_t \leq \epsilon t$. Denote by τ the stopping time $\tau = \min\{t : N_t \leq n - 15\epsilon n\}$. By definition of η_t we have

$$\mathbf{E}\,[\xi_t \mid \mathcal{F}_{t-1}] = pN_{t-1} - p\mathbf{1}_{\{A_{t-1}=0\}} - 1\,.$$

As $\{N_t\}$ is a decreasing sequence, we deduce that as long as $t < \tau$, we have $\mathbf{E}\,[\xi_t \mid \mathcal{F}_{t-1}] > -D\epsilon$ for $D > 0$ large enough. Hence, the process $\{D\epsilon j - Y_j\}_{j=0}^{t\wedge\tau}$ is a submartingale for any t. By Doob's maximal L^2 inequality we have

$$\mathbf{E}\,[\max_{j\leq t\wedge\tau}(D\epsilon j - Y_j)^2] \leq 4\mathbf{E}\,[(D\epsilon(t\wedge\tau) - Y_{t\wedge\tau})^2]\,. \tag{11}$$

For any $j < \tau$ the random variable η_j can be stochastically bounded from below by a $\text{Bin}(n - 15\epsilon n, p)$ random variable and above by a $\text{Bin}(n,p)$ random variable. Hence for any $k < j < \tau$ we have

$$\Big|\mathbf{E}\,[\xi_j - D\epsilon \mid \mathcal{F}_k]\Big| = O(\epsilon)\,,$$

and so

$$\mathbf{E}\,[(\xi_j - D\epsilon)(\xi_k - D\epsilon)] = O(\epsilon^2)\,.$$

We conclude that as long as $t < \tau$

$$\mathbf{E}\,[(D\epsilon t - Y_t)^2] \leq 2\sum_{k<j}^{t} \mathbf{E}\,[(\xi_j - D\epsilon)(\xi_k - D\epsilon)] + \sum_{j=1}^{t} \mathbf{E}\,[(\xi_j - D\epsilon)^2] = O(\epsilon^2 t^2 + t)\,.$$

Lemma 3 implies that for n large enough,

$$\mathbf{P}\Big(N_T \leq n - 15\epsilon n\Big) \leq e^{-3c\epsilon n} \leq \frac{1}{n^2}\,, \tag{12}$$

and as $\{N_t\}$ is a decreasing sequence we deduce that $\mathbf{P}(\tau \leq T) \leq n^{-2}$. Hence for any $t \leq T$

$$\begin{aligned} \mathbf{E}\,[(D\epsilon t - Y_t)^2] &\leq \mathbf{E}\,[(D\epsilon(t \wedge \tau) - Y_{t\wedge\tau})^2 \mathbf{1}_{\{t<\tau\}}] + O(n^2)\mathbf{P}(t \geq \tau) \\ &= O(\epsilon^2 t^2 + t)\,. \end{aligned}$$

We deduce by (11) and Jensen inequality that for any $t \leq T$

$$\mathbf{E}\,[\min_{j\leq t}(Y_j - D\epsilon j)] = O(\epsilon t + \sqrt{t})\,,$$

hence $\mathbf{E}\,[\min_{j\leq t} Y_j] = O(\epsilon t + \sqrt{t})$ and so by (4) we obtain (9). Inequality (10) follows immediately from the relation $Z_t = A_t - Y_t$. $\square$

Lemma 7 *If* $p = \frac{1+\epsilon}{n}$ *then for all* $t < 3\epsilon(n)n$

$$\mathbf{E}\,N_t = n(1-p)^t + O(\epsilon^2 n)\,, \tag{13}$$

and

$$\mathbf{E}\,\xi_t = \epsilon - \frac{t}{n} + O(\epsilon^2)\,. \tag{14}$$

PROOF. Observe that by (1) we have that

$$\mathbf{E}\,[N_t \mid \mathcal{F}_{t-1}] = (1-p)N_{t-1} - (1-p)\mathbf{1}_{\{A_{t-1}=0\}}\,.$$

By iterating this relation we get that $\mathbf{E}\,N_t = n(1-p)^t + O(\mathbf{E}\,Z_t)$ which by Lemma 6 yields (13) (observe that for $t = 3\epsilon n$ we have $\epsilon t > \sqrt{t}$ by our assumption on ϵ). Since

$$E[\xi_t \mid \mathcal{F}_{t-1}] = pN_{t-1} - p\mathbf{1}_{\{A_{t-1}=0\}} - 1\,,$$

by taking expectations and using (13) we get

$$\begin{aligned}\mathbf{E}\,\xi_t &= (1+\epsilon)(1-\frac{1+\epsilon}{n})^t - 1 + O(\epsilon^2)\\ &= (1+\epsilon)(1-(1+\epsilon)t/n) - 1 + O(\epsilon^2) = \epsilon - \frac{t}{n} + O(\epsilon^2)\,,\end{aligned}$$

where we used the fact that $(1-x)^t = 1 - tx + O(t^2x^2)$. □

Proof of Theorem 2. Write $T = 3\epsilon n$ and $\xi_j^* = \mathbf{E}\,[\xi_j \mid \mathcal{F}_{j-1}]$. The process

$$M_t = Y_t - \sum_{j=1}^{t} \xi_j^*\,,$$

is a martingale. By Doob's maximal L^2 inequality we have that

$$\mathbf{E}\,(\max_{t\le T} M_t^2)) \le 4\mathbf{E}\,M_T^2\,.$$

As M_t has orthogonal increments with bounded second moment we conclude that $\mathbf{E}\,M_T^2 = O(T)$, hence, by Jensen's inequality we have

$$\mathbf{E}\Big[\max_{t\le T}\Big|Y_t - \sum_{j=1}^{t}\xi_j^*\Big|\Big] \le O(\sqrt{T}) = O(\sqrt{\epsilon n})\,. \tag{15}$$

As $\xi_j^* = pN_{j-1} - p\mathbf{1}_{\{A_{j-1}=0\}} - 1$ by (3) we have

$$\mathbf{E}\,|\xi_j^* - \mathbf{E}\,\xi_j| = p\mathbf{E}\,|A_{j-1} + \mathbf{1}_{\{A_{j-1}=0\}} - \mathbf{E}\,A_{j-1} - \mathbf{E}\,\mathbf{1}_{\{A_{j-1}=0\}}|\,.$$

By the triangle inequality and Lemma 6 we conclude that for all $j \le T$

$$\mathbf{E}\,|\xi_j^* - \mathbf{E}\,\xi_j| \le p\cdot O(\epsilon j + \sqrt{j})\,,$$

and hence for any $t \le T$

$$\mathbf{E}\Big[\sum_{j\le t}|\xi_j^* - \mathbf{E}\,\xi_j|\Big] \le p\cdot O(\epsilon t^2 + t^{3/2}) \le O(\epsilon^3 n)\,.$$

By the triangle inequality we get

$$\mathbf{E}\Big[\max_{t\le T}\Big|\sum_{j=1}^{t}(\xi_j^* - \mathbf{E}\,\xi_j)\Big|\Big] \le O(\epsilon^3 n)\,. \tag{16}$$

Using the triangle inequality, (15), (16) and Markov inequality gives

$$\mathbf{P}\Big(\max_{t\le T}\Big|Y_t-\sum_{j=1}^{t}\mathbf{E}\,\xi_j\Big|\ge\delta\epsilon^2 n\Big)\le\delta^{-1}(O(\epsilon)+O((\epsilon^3 n)^{-1/2}))\longrightarrow 0\,. \quad (17)$$

Lemma 7 implies that for any $b>0$

$$\sum_{j=1}^{b\epsilon n}\mathbf{E}\,\xi_j=\sum_{j=1}^{b\epsilon n}\Big(\epsilon-\frac{t}{n}+O(\epsilon^2)\Big)=(b-\frac{b^2}{2})\epsilon^2 n+O(\epsilon^3 n)\,. \quad (18)$$

By (17) and (18) we deduce that for $\eta>0$ small enough, with probability tending to 1, the process Y_t is strictly positive at times $[\eta\epsilon n,(2-\eta)\epsilon n]$ and hence

$$\mathbf{P}\Big(|\mathcal{C}_1|>2(1-\eta)\epsilon n\Big)\ge 1-O\Big(\eta^{-1}(\epsilon+(\epsilon^3 n)^{-1/2})\Big)\,.$$

We also deduce by (17) and (18) that at time $t=(2+\eta)\epsilon n$ we have $Y_t\le -\frac{\eta^2}{3}\epsilon^2 n$ and at all times $t<\eta\epsilon n$ we have that $Y_t>-\frac{\eta^2}{3}\epsilon^2 n$ with probability tending to 1. As component sizes are excursion lengths of Y_t above its past minima, we conclude that by time $2(1+\eta)\epsilon n$ we have explored completely at least one component of size at least $2(1-\eta)\epsilon n$. As $N_t\le n-t$ for $t>2(1-\eta)\epsilon n$ and $\eta<1/4$ we have $\mathbf{E}\,[\eta_t-1\mid\mathcal{F}_{t-1}]\le-\frac{\epsilon}{2}$. By optional stopping we immediately get that if $t_j>2(1-\eta)\epsilon n$ then $\mathbf{E}\,[t_{j+1}-t_j]\le 2\epsilon^{-1}$. Thus if $\mathcal{C}$ is a component which we began discovering after time $2(1-\eta)\epsilon n$ we have

$$\mathbf{P}(|\mathcal{C}|\ge\epsilon n)\le\frac{2}{\epsilon^2 n}\,.$$

Denote by $X(\eta)$ the number of vertices of which $|C(v)|>\epsilon n$ which we began discovering after time $2(1-\eta)\epsilon n$, then we learn that $\mathbf{E}\,X(\eta)\le\frac{2}{\epsilon^2}$. Denote by $\mathcal{C}_1(\eta)$ the largest component which we began discovering after time $2(1-\eta)\epsilon n$. Clearly if $|\mathcal{C}_1(\eta)|>\epsilon n$ then $X(\eta)>\epsilon n$, thus by Markov inequality

$$\mathbf{P}(|\mathcal{C}_1(\eta)|>\epsilon n)\le\frac{2}{\epsilon^3 n}\to 0\,.$$

Thus we have proved that there exists a unique component of size between $2(1-\eta)\epsilon n$ and $2(1+\eta)\epsilon n$. Condition on this event and consider the graph remained on the complement of this component. This graph has m vertices where

$$|m-(n-2\epsilon n)|<2\eta\epsilon n\,,$$

and as $p = \frac{1+\epsilon}{n}$ we have that

$$\Big|p - \Big(1 - \frac{\epsilon}{m}\Big)\Big| \leq \frac{2\eta\epsilon + O(\epsilon^2)}{m}\,.$$

This graph is distributed as $G(m,p)$ restricted to the event that it does not contain a component of size between $2(1-\eta)\epsilon n$ and $2(1+\eta)\epsilon n$. By Theorem 1 we know that the event that there exists such a component has probability $o(1)$. Thus for any collection of graphs $\mathcal{B}$ on m vertices which does not contain such a component, the probability of $\mathcal{B}$ in the remaining graph is $(1+o(1))\mathbf{P}_{m,p}(\mathcal{B})$ where $\mathbf{P}_{m,p}$ is the usual $G(m,p)$ probability measure. Thus, we conclude by Theorem 1 that for any integer $\ell > 1$ and $\eta' > 0$

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon^{-2}(n)\log(n\epsilon^3)} - 1\Big| > \eta'\Big) \to 0\,,$$

concluding the proof of the theorem. □

Remark. With a little more effort it is possible to show for the supercritical case, that in the exploration process for any fixed ℓ, the ℓ-th largest component is discovered *after* the largest component is discovered.

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