

# UNIQUENESS OF ROOTS UP TO CONJUGACY FOR SOME AFFINE AND FINITE TYPE ARTIN GROUPS

EON-KYUNG LEE AND SANG-JIN LEE

ABSTRACT. Let  $G$  be one of the Artin groups of finite type  $\mathbf{B}_n = \mathbf{C}_n$  and affine type  $\tilde{\mathbf{A}}_{n-1}$ ,  $\tilde{\mathbf{C}}_{n-1}$ . In this paper, we show that if  $\alpha$  and  $\beta$  are elements of  $G$  such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ , then  $\alpha$  and  $\beta$  are conjugate in  $G$ . For the Artin group of type  $\mathbf{A}_n$ , this was recently proved by J. González-Meneses.

In fact, we prove a stronger theorem, from which the above result follows easily by using descriptions of those Artin groups as subgroups of the braid group on  $n+1$  strands. Let  $P$  be a subset of  $\{1, \dots, n\}$ . An  $n$ -braid is said to be  $P$ -pure if its induced permutation fixes each  $i \in P$ , and  $P$ -straight if it is  $P$ -pure and it becomes trivial when we delete all the  $i$ -th strands for  $i \notin P$ . Exploiting the Nielsen-Thurston classification of braids, we show that if  $\alpha$  and  $\beta$  are  $P$ -pure  $n$ -braids such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ , then there exists a  $P$ -straight  $n$ -braid  $\gamma$  with  $\beta = \gamma\alpha\gamma^{-1}$ . Moreover, if  $1 \in P$ , the conjugating element  $\gamma$  can be chosen to have the first strand algebraically unlinked with the other strands. Especially in case of  $P = \{1, \dots, n\}$ , our result implies the uniqueness of roots of pure braids, which was known by V. G. Bardakov and by D. Kim and D. Rolfsen.

*Keywords:* Artin group, braid group, uniqueness of roots

*MSC:* 20F36, 20F10

## 1. INTRODUCTION

Let  $M$  be a symmetric  $n \times n$  matrix with integer entries  $m_{ij} \in \mathbb{N} \cup \{\infty\}$  where  $m_{ii} = 1$  and  $m_{ij} \geq 2$  for  $i \neq j$ . The Artin group of type  $M$  is defined by the presentation

$$A(M) = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \text{ for all } i \neq j, m_{ij} \neq \infty \rangle.$$

The Coxeter group  $W(M)$  of type  $M$  is the quotient of  $A(M)$  by the relation  $s_i^2 = 1$ . We say that  $A(M)$  is of *finite type* if the associated Coxeter group  $W(M)$  is finite, and that  $A(M)$  is of *affine (or Euclidean) type* if  $W(M)$  acts as a proper, cocompact group of isometries on some Euclidean space with the generators  $s_1, \dots, s_n$  acting as affine reflections. It is convenient to define an Artin group by a *Coxeter graph*, whose vertices are numbered  $1, \dots, n$  and which has an edge labelled  $m_{ij}$  between the vertices  $i$  and  $j$  whenever  $m_{ij} \geq 3$  or  $m_{ij} = \infty$ . The label 3 is usually suppressed.

In this paper, we show the uniqueness of roots up to conjugacy for elements of the Artin groups of finite type  $\mathbf{A}_n$ ,  $\mathbf{B}_n = \mathbf{C}_n$  and affine type  $\tilde{\mathbf{A}}_{n-1}$ ,  $\tilde{\mathbf{C}}_{n-1}$ . The Coxeter graphs associated to them are in Figure 1.

**Theorem 1.1.** *Let  $G$  denote one of the Artin groups of finite type  $\mathbf{A}_n$ ,  $\mathbf{B}_n = \mathbf{C}_n$  and affine type  $\tilde{\mathbf{A}}_{n-1}$ ,  $\tilde{\mathbf{C}}_{n-1}$ . If  $\alpha, \beta \in G$  are such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ , then  $\alpha$  and  $\beta$  are conjugate in  $G$ .*

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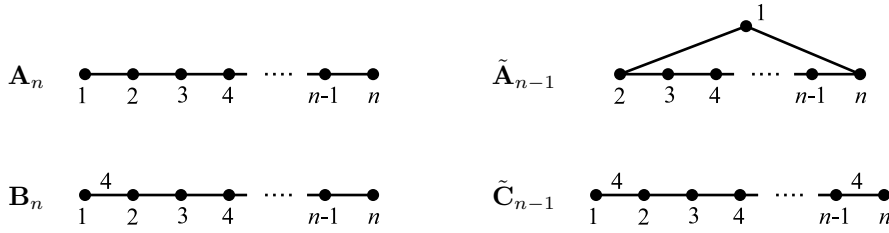


FIGURE 1. Coxeter graphs

In fact, we prove a stronger theorem. Before stating it, let us explain the motivations. The Artin group  $A(\mathbf{A}_n)$  is well-known as  $B_{n+1}$ , the braid group on  $n+1$  strands. The generators of  $B_{n+1}$  are usually written as  $\sigma_i$ , hence it has the presentation

$$B_{n+1} = \left\langle \sigma_1, \dots, \sigma_n \mid \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i-j| = 1. \end{array} \right\rangle.$$

The following are well-known theorems on uniqueness of roots of braids.

**Theorem 1.2** (J. González-Meneses [Gon03]). *Let  $\alpha$  and  $\beta$  be  $n$ -braids such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ . Then  $\alpha$  and  $\beta$  are conjugate in  $B_n$ .*

**Theorem 1.3** (V. G. Bardakov [Bar92], D. Kim and D. Rolfsen [KR03]). *Let  $\alpha$  and  $\beta$  be pure braids such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ . Then  $\alpha$  and  $\beta$  are equal.*

Theorem 1.2 was conjectured by G. S. Makanin [Mak71] in the early seventies, and proved recently by J. González-Meneses. Thus the new contribution of Theorem 1.1 is for the Artin groups of type  $\mathbf{B}_n = \mathbf{C}_n$ ,  $\tilde{\mathbf{A}}_{n-1}$  and  $\tilde{\mathbf{C}}_{n-1}$ . Theorem 1.3 was first proved by V. G. Bardakov by combinatorial arguments, and it follows easily from the bi-orderability of pure braids by D. Kim and D. Rolfsen. (To see this, let  $<$  be a bi-ordering of pure braids. If  $\alpha > \beta$  (resp.  $\alpha < \beta$ ), then  $\alpha^k > \beta^k$  (resp.  $\alpha^k < \beta^k$ ) for all  $k \geq 1$ . Therefore,  $\alpha^k = \beta^k$  implies  $\alpha = \beta$ .)

It is worth mentioning that D. Bessis showed the uniqueness of roots up to conjugacy for periodic elements in the braid groups of irreducible well-generated complex reflection groups, and hence for periodic elements in finite type Artin groups: if  $G$  is the braid group of an irreducible well-generated complex reflection group and if  $\alpha, \beta \in G$  are such that  $\alpha$  has a central power and  $\alpha^k = \beta^k$  for some nonzero integer  $k$ , then  $\alpha$  and  $\beta$  are conjugate in  $G$  [Bes06, Theorem 12.5 (ii)].

Comparing the above two theorems, we can see that one obtains a stronger result for pure braids. Motivated by the above observation, we study the case of “partially pure” braids—that is, braids some of whose strands are pure. Moreover, the Artin groups  $A(\mathbf{B}_n)$ ,  $A(\tilde{\mathbf{A}}_{n-1})$  and  $A(\tilde{\mathbf{C}}_{n-1})$  are isomorphic to some subgroups of  $B_{n+1}$ , which can be described by pure strands and linking number of the first strand with the other strands. In order to deal with “partially pure” braids and elements of those Artin groups simultaneously, we introduce the following definitions.

**Definition 1.4.** For an  $n$ -braid  $\alpha$ , let  $\pi_\alpha$  denote the induced permutation of  $\alpha$ .

- For an  $n$ -braid  $\alpha$  and an integer  $1 \leq i \leq n$ , we say that  $\alpha$  is  $i$ -pure, or the  $i$ -th strand of  $\alpha$  is pure, if  $\pi_\alpha(i) = i$ .
- Let  $B_{n,1}$  denote the subgroup of  $B_n$  consisting of 1-pure braids.
- Let  $P \subset \{1, \dots, n\}$ . An  $n$ -braid  $\alpha$  is said to be  $P$ -pure if  $\alpha$  is  $i$ -pure for each  $i \in P$ . Note that  $\{1, \dots, n\}$ -pure braids are nothing more than pure braids in the usual sense.

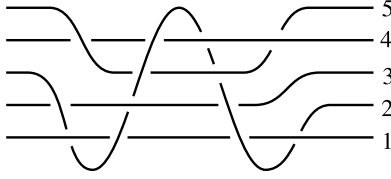


FIGURE 2. This braid is  $\{1, 4, 5\}$ -pure,  $\{1, 4\}$ -straight,  $\{1, 5\}$ -straight and 1-unlinked.

- Let  $P \subset \{1, \dots, n\}$ . An  $n$ -braid is said to be  $P$ -straight if it is  $P$ -pure and it becomes trivial when we delete all the  $i$ -th strands for  $i \notin P$ . Note the following: if  $|P| = 1$ , then a braid is  $P$ -pure if and only if it is  $P$ -straight; if  $|P| = n$  and  $\alpha$  is a  $P$ -straight  $n$ -braid, then  $\alpha$  is the identity; a braid  $\alpha$  is called a *brunnian braid* if it is  $P$ -straight for all  $P$  with  $|P| = n - 1$ .

For example, the braid in Figure 2 is  $\{1, 4, 5\}$ -pure,  $\{1, 4\}$ -straight and  $\{1, 5\}$ -straight.

**Definition 1.5.** There is a homomorphism  $\text{lk} : B_{n,1} \rightarrow \mathbb{Z}$  which measures the linking number of the first strand with the other strands: let  $\sigma_1, \dots, \sigma_{n-1}$  be the Artin generators for  $B_n$ , then  $B_{n,1}$  is generated by  $\sigma_1^2, \sigma_2, \sigma_3, \dots, \sigma_{n-1}$ , and the homomorphism  $\text{lk}$  is defined by  $\text{lk}(\sigma_1^2) = 1$  and  $\text{lk}(\sigma_i) = 0$  for  $i \geq 2$ . Note that  $\text{lk}(\cdot)$  is a conjugacy invariant in  $B_{n,1}$  because  $\text{lk}(\gamma\alpha\gamma^{-1}) = \text{lk}(\gamma) + \text{lk}(\alpha) - \text{lk}(\gamma) = \text{lk}(\alpha)$  for any  $\alpha, \gamma \in B_{n,1}$ . A braid  $\alpha$  is said to be *1-unlinked* if it is 1-pure and  $\text{lk}(\alpha) = 0$ . For example, the braid in Figure 2 is 1-unlinked.

It is well known that the following isomorphisms hold [Cri99, All02, CC05, BM05]:

$$\begin{aligned} A(\mathbf{B}_n) &\simeq B_{n+1,1}; \\ A(\tilde{\mathbf{A}}_{n-1}) &\simeq \{\alpha \in B_{n+1,1} \mid \alpha \text{ is 1-unlinked}\}; \\ A(\tilde{\mathbf{C}}_{n-1}) &\simeq \{\alpha \in B_{n+1,1} \mid \alpha \text{ is } \{1, n+1\}\text{-pure}\}. \end{aligned}$$

As we do not need to consider the Artin group  $A(\mathbf{A}_n)$  due to Theorem 1.2, it suffices to consider 1-pure braids. From now on, we restrict ourselves to  $B_{n,1}$ , the group of 1-pure braids on  $n$  strands. Exploiting the Nielsen-Thurston classification of braids, we establish the following theorem.

**Theorem 1.6.** *Let  $P$  be a subset of  $\{1, \dots, n\}$  with  $1 \in P$ . Let  $\alpha$  and  $\beta$  be  $P$ -pure  $n$ -braids such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ . Then there exists a  $P$ -straight, 1-unlinked  $n$ -braid  $\gamma$  with  $\beta = \gamma\alpha\gamma^{-1}$ .*

Applying Theorem 1.6 to  $\{1\}$ -pure  $(n+1)$ -braids (resp.  $\{1, n+1\}$ -pure  $(n+1)$ -braids), we have Theorem 1.1 for  $A(\mathbf{B}_n)$  and  $A(\tilde{\mathbf{A}}_{n-1})$  (resp. for  $A(\tilde{\mathbf{C}}_{n-1})$ ).

We close this section with some remarks. An easy consequence of Theorem 1.1 is the following.

Let  $G$  denote one of the Artin groups of finite type  $\mathbf{A}_n$ ,  $\mathbf{B}_n = \mathbf{C}_n$  and affine type  $\tilde{\mathbf{A}}_{n-1}$ ,  $\tilde{\mathbf{C}}_{n-1}$ . Let  $\alpha, \beta \in G$  and let  $k$  be a nonzero integer. Then  $\alpha$  is conjugate to  $\beta$  if and only if  $\alpha^k$  is conjugate to  $\beta^k$ .

Theorem 1.3 follows easily from Theorem 1.6: Let  $\alpha$  and  $\beta$  be pure  $n$ -braids with  $\alpha^k = \beta^k$ . In our terminology, both  $\alpha$  and  $\beta$  are  $\{1, \dots, n\}$ -pure, hence there exists a  $\{1, \dots, n\}$ -straight  $n$ -braid  $\gamma$  such that  $\beta = \gamma\alpha\gamma^{-1}$ . Because  $\gamma$  is  $\{1, \dots, n\}$ -straight, we have  $\gamma = 1$ , hence  $\alpha = \beta$ .

Theorem 1.1, even for  $A(\mathbf{B}_n)$ , does not follow easily from Theorem 1.2 because there are 1-pure braids that are conjugate in  $B_n$ , but not in the subgroup  $B_{n,1}$ , as in the following example.

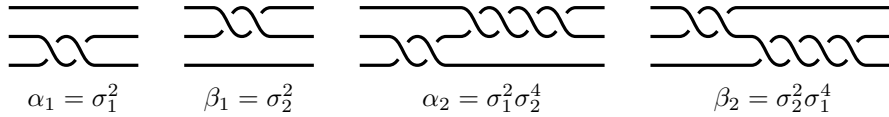


FIGURE 3.  $\alpha_i, \beta_i \in B_{n,1}$  are conjugate in  $B_n$  but not in  $B_{n,1}$ .

**Example 1.7.** Consider the 1-pure 3-braids which are depicted in Figure 3:

$$\begin{cases} \alpha_1 = \sigma_1^2, \\ \beta_1 = \sigma_2^2, \end{cases} \quad \text{and} \quad \begin{cases} \alpha_2 = \sigma_1^2 \sigma_2^4, \\ \beta_2 = \sigma_2^2 \sigma_1^4. \end{cases}$$

Because  $\Delta \alpha_i \Delta^{-1} = \beta_i$  for  $i = 1, 2$ , where  $\Delta = \sigma_1 \sigma_2 \sigma_1$ , the braid  $\alpha_i$  is conjugate to  $\beta_i$  in  $B_3$ . However,  $\alpha_i$  is not conjugate to  $\beta_i$  in  $B_{3,1}$  for  $i = 1, 2$  because  $\text{lk}(\alpha_1) = \text{lk}(\alpha_2) = 1$ ,  $\text{lk}(\beta_1) = 0$  and  $\text{lk}(\beta_2) = 2$ . Note that  $\alpha_1$  and  $\beta_1$  are reducible, and that  $\alpha_2$  and  $\beta_2$  are pseudo-Anosov.

## 2. PRELIMINARIES

Here, we review basic definitions and results on braids. See [Art25, Bir74, Thu88, FLP79, LL08]. Let  $D^2 = \{z \in \mathbb{C} : |z| \leq n+1\}$ , and let  $D_n$  be the  $n$ -punctured disk  $D^2 \setminus \{1, 2, \dots, n\}$ . The Artin braid group  $B_n$  is the group of automorphisms of  $D_n$  that fix the boundary pointwise, modulo isotopy relative to the boundary. Geometrically, an  $n$ -braid can be interpreted as an isotopy class of the collections of pairwise disjoint  $n$  strands  $l = l_1 \cup \dots \cup l_n \subset D^2 \times [0, 1]$  such that  $l \cap (D^2 \times \{t\})$  consists of  $n$  points for each  $t \in [0, 1]$ , and, in particular, it is  $\{(1, t), \dots, (n, t)\}$  for  $t \in \{0, 1\}$ . The admissible isotopies lie in the interior of  $D^2 \times [0, 1]$ . The center of the  $n$ -braid group  $B_n$  is infinite cyclic generated by  $\Delta^2$ , where  $\Delta = \sigma_1(\sigma_2 \sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_1)$ .

The well-known Nielsen-Thurston classification of mapping classes of punctured surfaces into periodic, reducible and pseudo-Anosov ones [Thu88, FLP79] yields an analogous classification of braids: an  $n$ -braid  $\alpha$  is *periodic* if some power of  $\alpha$  is central;  $\alpha$  is *reducible* if there exists an essential curve system in  $D_n$  which is invariant up to isotopy under the action of  $\alpha$ ;  $\alpha$  is *pseudo-Anosov* if no non-trivial power of  $\alpha$  is reducible.

**Lemma 2.1.** *Let  $\alpha, \beta \in B_n$  be such that  $\alpha^k = \beta^k$  for a nonzero integer  $k$ . Then*

- (i)  $\alpha$  and  $\beta$  are of the same Nielsen-Thurston type;
- (ii) if  $\alpha$  is pseudo-Anosov, then  $\alpha = \beta$ .

*Proof.* (i) is well known. (ii) was proved by González-Meneses [Gon03]. □

**2.1. Periodic braids.** Let  $\delta = \sigma_{n-1} \cdots \sigma_1$  and  $\epsilon = \delta \sigma_1$ , then  $\delta^n = \Delta^2 = \epsilon^{n-1}$ . (If we need to specify the number of strands, we will write  $\delta = \delta_{(n)}$ ,  $\epsilon = \epsilon_{(n)}$  and  $\Delta = \Delta_{(n)}$ .) Note that  $\delta$  and  $\epsilon$  are represented by rigid rotations of the  $n$ -punctured disk as in Figure 4 when the punctures are at the center of the disk or on a round circle centered at the origin. By Brouwer, Kerékjártó and Eilenberg, it is known that an  $n$ -braid  $\alpha$  is periodic if and only if it is conjugate to a power of either  $\delta$  or  $\epsilon$  [Bro19, Ker19, Eil34, BDM02].

**Lemma 2.2.** *An  $n$ -braid  $\alpha$  is periodic if and only if  $\alpha$  is conjugate to either  $\delta^m$  or  $\epsilon^m$  for some integer  $m$ . Further, if  $\alpha$  is periodic and non-central, then exactly one of the following holds.*

- (i)  $\alpha$  is conjugate to  $\delta^m$  for some  $m \not\equiv 0 \pmod{n}$ . In this case,  $\alpha$  has no pure strand.
- (ii)  $\alpha$  is conjugate to  $\epsilon^m$  for some  $m \not\equiv 0 \pmod{n-1}$ . In this case,  $\alpha$  has only one pure strand.

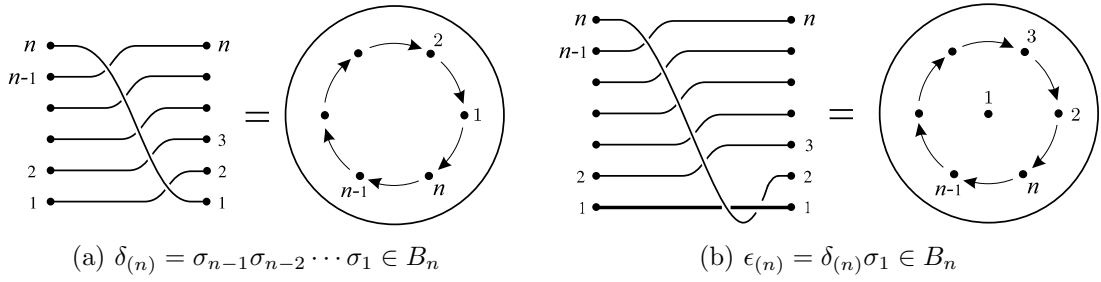


FIGURE 4. The braid  $\delta_{(n)}$  is represented by the  $2\pi/n$ -rotation of the  $n$ -punctured disk in a clockwise direction where the punctures lie on a round circle as in (a). The braid  $\epsilon_{(n)}$  is represented by the  $2\pi/(n-1)$ -rotation of the  $n$ -punctured disk in a clockwise direction where one puncture is at the center and the other  $n-1$  punctures lie on a round circle as in (b).

**Corollary 2.3.** *Let  $\alpha$  be a periodic  $n$ -braid whose first strand is pure.*

- (i) *If  $\alpha$  has at least two pure strands, then  $\alpha$  is central.*
- (ii) *If  $\alpha$  is 1-unlinked, then  $\alpha$  is the identity.*

*Proof.* (i) It is immediate from Lemma 2.2.

(ii) Let  $\alpha$  be 1-unlinked and periodic. Because  $\alpha$  is 1-pure, it is conjugate to  $\epsilon^m$ . Because  $\text{lk}(\epsilon) = 1$  and  $\alpha$  is 1-unlinked,  $0 = \text{lk}(\alpha) = \text{lk}(\epsilon^m) = m \text{lk}(\epsilon) = m$ , hence  $\alpha$  is the identity.  $\square$

## 2.2. Reducible braids.

**Definition 2.4.** A curve system  $\mathcal{C}$  in  $D_n$  means a finite collection of disjoint simple closed curves in  $D_n$ . It is said to be *essential* if each component is homotopic neither to a point nor to a puncture nor to the boundary. It is said to be *unnnested* if none of its components encloses another component as in Figure 5 (b).

**Definition 2.5.** The  $n$ -braid group  $B_n$  acts on the set of curve systems in  $D_n$ . Let  $\alpha * \mathcal{C}$  denote the left action of  $\alpha \in B_n$  on the curve system  $\mathcal{C}$  in  $D_n$ . An  $n$ -braid  $\alpha$  is said to be *reducible* if  $\alpha * \mathcal{C} = \mathcal{C}$  for some essential curve system  $\mathcal{C}$  in  $D_n$ . Such a curve system  $\mathcal{C}$  is called a *reduction system* of  $\alpha$ .

**2.2.1. Canonical reduction system.** For a reduction system  $\mathcal{C}$  of an  $n$ -braid  $\alpha$ , let  $D_{\mathcal{C}}$  be the closure of  $D_n \setminus N(\mathcal{C})$  in  $D_n$ , where  $N(\mathcal{C})$  is a regular neighborhood of  $\mathcal{C}$ . The restriction of  $\alpha$  induces an automorphism on  $D_{\mathcal{C}}$  that is well defined up to isotopy. Due to Birman, Lubotzky and McCarthy [BLM83] and Ivanov [Iva92], for any  $n$ -braid  $\alpha$ , there is a unique *canonical reduction system*  $\mathcal{R}(\alpha)$  with the following properties.

- (i)  $\mathcal{R}(\alpha^m) = \mathcal{R}(\alpha)$  for all  $m \neq 0$ .
- (ii)  $\mathcal{R}(\beta\alpha\beta^{-1}) = \beta * \mathcal{R}(\alpha)$  for all  $\beta \in B_n$ .
- (iii) The restriction of  $\alpha$  to each component of  $D_{\mathcal{R}(\alpha)}$  is either periodic or pseudo-Anosov. A reduction system with this property is said to be *adequate*.
- (iv) If  $\mathcal{C}$  is an adequate reduction system of  $\alpha$ , then  $\mathcal{R}(\alpha) \subset \mathcal{C}$ .

By the properties of canonical reduction systems, a braid  $\alpha$  is reducible and non-periodic if and only if  $\mathcal{R}(\alpha) \neq \emptyset$ . Let  $\mathcal{R}_{\text{ext}}(\alpha)$  denote the collection of the outermost components of  $\mathcal{R}(\alpha)$ . Then  $\mathcal{R}_{\text{ext}}(\alpha)$  is an unnnested curve system satisfying the properties (i) and (ii).

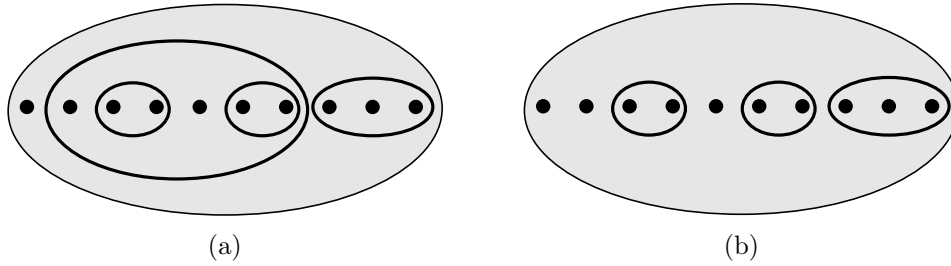


FIGURE 5. (a) shows a standard curve system in  $D_{10}$ . (b) shows the unnested standard curve system  $\mathcal{C}_{\mathbf{n}}$  for  $\mathbf{n} = (1, 1, 2, 1, 2, 3)$

2.2.2. *Standard reduction system.* In this paper we use a notation, introduced in [LL08], for reducible braids with standard reduction system.

**Definition 2.6.** An essential curve system in  $D_n$  is said to be *standard* if each component is isotopic to a round circle centered at the real axis as in Figure 5 (a).

The unnested standard curve systems in  $D_n$  are in one-to-one correspondence with the  $r$ -compositions of  $n$  for  $2 \leq r \leq n - 1$ . Recall that an ordered  $r$ -tuple  $\mathbf{n} = (n_1, \dots, n_r)$  is an  $r$ -composition of  $n$  if  $n_i \geq 1$  for each  $i$  and  $n = n_1 + \dots + n_r$ .

**Definition 2.7.** For a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$ , let  $\mathcal{C}_{\mathbf{n}}$  denote the unnested standard curve system  $\cup_{n_i \geq 2} \mathcal{C}_i$ , where each  $\mathcal{C}_i$  is a round circle, centered at the real line, enclosing the punctures  $\{m \mid \sum_{j=1}^{i-1} n_j < m \leq \sum_{j=1}^i n_j\}$ . For example, Figure 5 (b) shows the unnested standard curve system  $\mathcal{C}_{\mathbf{n}}$  for  $\mathbf{n} = (1, 1, 2, 1, 2, 3)$ .

The  $r$ -braid group  $B_r$  acts on the set of  $r$ -compositions of  $n$  via the induced permutations: for an  $r$ -composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$  and  $\alpha \in B_r$  with induced permutation  $\theta$ ,  $\alpha * \mathbf{n} = (n_{\theta^{-1}(1)}, \dots, n_{\theta^{-1}(r)})$ .

**Remark.** Throughout this paper, braids and permutations act on the left. That is, if  $\alpha$  and  $\beta$  are  $n$ -braids, then  $(\alpha\beta) * \mathcal{C} = \alpha * (\beta * \mathcal{C})$  for a curve system  $\mathcal{C}$  in  $D_n$ ; if  $\alpha$  and  $\beta$  are  $r$ -braids, then  $(\alpha\beta) * \mathbf{n} = \alpha * (\beta * \mathbf{n})$  for an  $r$ -composition  $\mathbf{n}$  of  $n$ ; if  $\pi_1$  and  $\pi_2$  are  $n$ -permutations, then  $(\pi_1 \circ \pi_2)(i) = \pi_1(\pi_2(i))$  for  $1 \leq i \leq n$ .

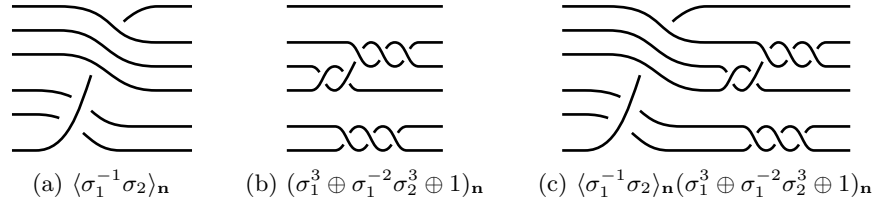
**Definition 2.8.** Let  $\mathbf{n} = (n_1, \dots, n_r)$  be a composition of  $n$ .

- Let  $\alpha_0 = l_1 \cup \dots \cup l_r$  be an  $r$ -braid with  $l_i \cap (D^2 \times \{1\}) = \{(i, 1)\}$  for each  $i$ . We define  $\langle \alpha_0 \rangle_{\mathbf{n}}$  as the  $n$ -braid obtained from  $\alpha_0$  by taking  $n_i$  parallel copies of  $l_i$  for each  $i$ . See Figure 6 (a).
- Let  $\alpha_i \in B_{n_i}$  for  $i = 1, \dots, r$ . We define  $(\alpha_1 \oplus \dots \oplus \alpha_r)_{\mathbf{n}}$  as the  $n$ -braid  $\alpha'_1 \alpha'_2 \dots \alpha'_r$ , where each  $\alpha'_i$  is the image of  $\alpha_i$  under the homomorphism  $B_{n_i} \rightarrow B_n$  defined by  $\sigma_j \mapsto \sigma_{n_1 + \dots + n_{i-1} + j}$ . See Figure 6 (b).

We will use the notation  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \dots \oplus \alpha_r)_{\mathbf{n}}$  throughout the paper. See Figure 6 (c). The following lemma shows some elementary properties.

**Lemma 2.9** ([LL08, Lemmas 3.5 and 3.6]). *Let  $\mathbf{n} = (n_1, \dots, n_r)$  be a composition of  $n$ .*

- (i) *The expression  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \dots \oplus \alpha_r)_{\mathbf{n}}$  is unique, i.e. if  $\langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \dots \oplus \alpha_r)_{\mathbf{n}} = \langle \beta_0 \rangle_{\mathbf{n}} (\beta_1 \oplus \dots \oplus \beta_r)_{\mathbf{n}}$ , then  $\alpha_i = \beta_i$  for  $i = 0, 1, \dots, r$ .*

FIGURE 6.  $\mathbf{n} = (2, 3, 1)$ 

- (ii) If  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}$ , then  $\alpha * \mathcal{C}_{\mathbf{n}}$  is standard and, further,  $\alpha * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\alpha_0 * \mathbf{n}}$ . Conversely, if  $\alpha * \mathcal{C}_{\mathbf{n}}$  is standard, then  $\alpha$  can be expressed as  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}$ .
- (iii)  $\langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}} = (\alpha_{\theta^{-1}(1)} \oplus \cdots \oplus \alpha_{\theta^{-1}(r)})_{\alpha_0 * \mathbf{n}} \langle \alpha_0 \rangle_{\mathbf{n}}$ , where  $\theta$  is the induced permutation of  $\alpha_0$ .
- (iv)  $\langle \alpha_0 \beta_0 \rangle_{\mathbf{n}} = \langle \alpha_0 \rangle_{\beta_0 * \mathbf{n}} \langle \beta_0 \rangle_{\mathbf{n}}$ .
- (v)  $(\langle \alpha_0 \rangle_{\mathbf{n}})^{-1} = \langle \alpha_0^{-1} \rangle_{\alpha_0 * \mathbf{n}}$ .
- (vi)  $(\alpha_1 \beta_1 \oplus \cdots \oplus \alpha_r \beta_r)_{\mathbf{n}} = (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}} (\beta_1 \oplus \cdots \oplus \beta_r)_{\mathbf{n}}$ .
- (vii)  $(\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}^{-1} = (\alpha_1^{-1} \oplus \cdots \oplus \alpha_r^{-1})_{\mathbf{n}}$ .
- (viii) Let  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}$ . Then  $\alpha = \Delta_{(n)}$  if and only if  $\alpha_0 = \Delta_{(r)}$  and  $\alpha_i = \Delta_{(n_i)}$  for  $1 \leq i \leq r$ .

### 2.3. Basic properties of $P$ -pure, $P$ -straight or 1-unlinked braids.

**Lemma 2.10.** Let  $\alpha$ ,  $\beta$  and  $\gamma$  be  $n$ -braids, and let  $P$  be a subset of  $\{1, \dots, n\}$ .

- (i) If  $\alpha$  is  $P$ -pure, then  $\gamma \alpha \gamma^{-1}$  is  $\pi_{\gamma}(P)$ -pure.
- (ii) If  $\alpha$  is  $P$ -straight, then  $\gamma \alpha \gamma^{-1}$  is  $\pi_{\gamma}(P)$ -straight.
- (iii) If  $\alpha$  is 1-unlinked and  $\gamma$  is 1-pure, then  $\gamma \alpha \gamma^{-1}$  is 1-unlinked.
- (iv) If both  $\alpha$  and  $\beta$  are  $P$ -pure (resp.  $P$ -straight, 1-unlinked), then  $\alpha^p \beta^q$  is  $P$ -pure (resp.  $P$ -straight, 1-unlinked) for any integers  $p$  and  $q$ .

*Proof.* (i) and (ii) are obvious.

(iii) It follows from  $\text{lk}(\gamma \alpha \gamma^{-1}) = \text{lk}(\gamma) + \text{lk}(\alpha) - \text{lk}(\gamma)$ .

(iv) It is obvious for  $P$ -purity and  $P$ -straightness. The 1-unlinkedness follows from  $\text{lk}(\alpha^p \beta^q) = p \text{lk}(\alpha) + q \text{lk}(\beta) = 0$ .  $\square$

**Lemma 2.11.** If  $\alpha \in B_{n,1}$  is a periodic braid, then there exists a 1-unlinked  $n$ -braid  $\gamma$  such that  $\gamma \alpha \gamma^{-1} = \epsilon^m$  for some integer  $m$ .

*Proof.* If  $\alpha$  is central, then we can take the identity as the conjugating element  $\gamma$ . Therefore we may assume that  $\alpha$  is non-central, hence  $\alpha$  is conjugate to  $\epsilon^m$  for some  $m \not\equiv 0 \pmod{n-1}$ . There exists an  $n$ -braid  $\gamma_1$  such that  $\gamma_1 \alpha \gamma_1^{-1} = \epsilon^m$ . Because  $\gamma_1 \alpha \gamma_1^{-1}$  is  $\pi_{\gamma_1}(1)$ -pure and  $\epsilon^m$  has the first strand as the only pure strand, we have  $\pi_{\gamma_1}(1) = 1$ , that is,  $\gamma_1$  is 1-pure. Let  $q = \text{lk}(\gamma_1)$  and  $\gamma = \epsilon^{-q} \gamma_1$ . Then

$$\gamma \alpha \gamma^{-1} = \epsilon^{-q} (\gamma_1 \alpha \gamma_1^{-1}) \epsilon^q = \epsilon^{-q} \epsilon^m \epsilon^q = \epsilon^m.$$

Since  $\gamma_1$  and  $\epsilon$  are 1-pure, so is  $\gamma$ . Since  $\text{lk}(\epsilon) = 1$ , we have

$$\text{lk}(\gamma) = \text{lk}(\gamma_1) + \text{lk}(\epsilon^{-q}) = \text{lk}(\gamma_1) - q = 0.$$

Therefore  $\gamma$  is a conjugating element from  $\alpha$  to  $\epsilon^m$ , which is 1-unlinked.  $\square$

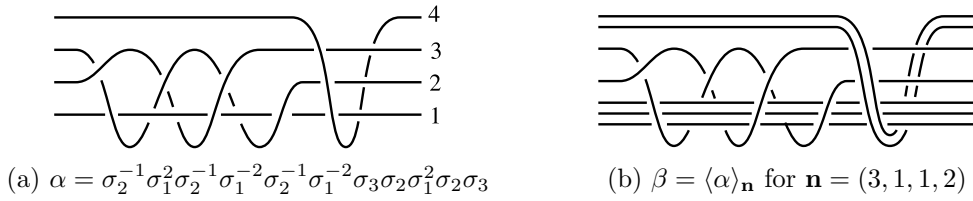


FIGURE 7. For the above 4-braid  $\alpha$ , we have  $\text{lk}_2(\alpha) = 0$ ,  $\text{lk}_3(\alpha) = -1$ ,  $\text{lk}_4(\alpha) = 1$ , hence  $\text{lk}(\alpha) = 0 + (-1) + 1 = 0$ . For the above 7-braid  $\beta$ , we have  $\text{lk}_2(\beta) = \text{lk}_3(\alpha) = 0$ ,  $\text{lk}_4(\beta) = 0$ ,  $\text{lk}_5(\beta) = -1$ ,  $\text{lk}_6(\beta) = \text{lk}_7(\alpha) = 1$ , hence  $\text{lk}(\beta) = 1$ .

**Definition 2.12.** For a braid  $\alpha \in B_{n,1}$  and an integer  $2 \leq i \leq n$ , we define the  $i$ -th linking number  $\text{lk}_i(\alpha)$  of  $\alpha$  as the linking number between the first and the  $i$ -th strands of  $\alpha$ . See Figure 7.

The following is an obvious relation between the linking number and the  $i$ -th linking number.

**Lemma 2.13.** Let  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}} \in B_{n,1}$  for a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$ . Then  $\text{lk}(\alpha) = \text{lk}(\alpha_1) + \sum_{i=2}^r n_i \text{lk}_i(\alpha_0)$ .

**Definition 2.14.** For a set  $P \subset \{1, 2, \dots, n\}$  and a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$ , define the sets  $P_{\mathbf{n},0}, P_{\mathbf{n},1}, \dots, P_{\mathbf{n},r}$  as follows:

$$\begin{aligned} P_{\mathbf{n},i} &= \{1 \leq j \leq n_i \mid (n_1 + \cdots + n_{i-1}) + j \in P\} \quad \text{for } i = 1, \dots, r; \\ P_{\mathbf{n},0} &= \{1 \leq i \leq r \mid P_{\mathbf{n},i} \neq \emptyset\}. \end{aligned}$$

Note that, using the above notations,  $P = \bigcup_{i=1}^r ((n_1 + \cdots + n_{i-1}) + P_{\mathbf{n},i})$ . The following lemma is easy.

**Lemma 2.15.** Let  $P \subset \{1, 2, \dots, n\}$  and  $\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}$  for a composition  $\mathbf{n}$  of  $n$ .

- (i)  $\alpha$  is  $P$ -pure if and only if  $\alpha_i$ 's are  $P_{\mathbf{n},i}$ -pure for all  $i = 0, 1, \dots, r$ .
- (ii)  $\alpha$  is  $P$ -straight if and only if  $\alpha_i$ 's are  $P_{\mathbf{n},i}$ -straight for all  $i = 0, 1, \dots, r$ .
- (iii) If  $\alpha_1$  is 1-unlinked, then  $(\alpha_1 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}}$  is 1-unlinked.

### 3. UNIQUENESS OF ROOTS UP TO CONJUGACY

In this section, we prove Theorem 1.6. Let us explain our strategy for proof. Suppose we are given  $P$ -pure braids  $\alpha$  and  $\beta$  such that  $\alpha^k = \beta^k$  for some nonzero integer  $k$ . Note that  $\alpha$  is either pseudo-Anosov, or periodic, or reducible and non-periodic. Lemma 3.2 deals with the case where  $\alpha$  is pseudo-Anosov or periodic. Now, suppose  $\alpha$  is reducible and non-periodic. There are three cases:  $\alpha_{\text{ext}}$  is pseudo-Anosov;  $\alpha_{\text{ext}}$  is central;  $\alpha_{\text{ext}}$  is periodic and non-central. (Here  $\alpha_{\text{ext}}$  is a particular tubular braid of  $\alpha$ . See Definition 3.3.) If  $\alpha_{\text{ext}}$  is either pseudo-Anosov or central, we may assume  $\alpha_{\text{ext}} = \beta_{\text{ext}}$ , and this case is resolved in Lemma 3.4. For the case where  $\alpha_{\text{ext}}$  is periodic and non-central, we construct a  $P$ -straight conjugating element from  $\alpha$  to  $\beta$ , and then modify this conjugating element in order to make it 1-unlinked. Lemma 3.6 is useful in this modification. In the end we give the proof of Theorem 1.6. Due to the lemmas mentioned above, it suffices to construct a  $P$ -straight conjugating element from  $\alpha$  to  $\beta$  for the case where  $\alpha_{\text{ext}}$  is periodic and non-central.

From now on, we will say that *Theorem 1.6 is true for  $(\alpha, \beta, P, k)$*  if  $(\alpha, \beta, P, k)$  is given as in Theorem 1.6 and there exists a  $P$ -straight, 1-unlinked braid  $\gamma$  with  $\beta = \gamma\alpha\gamma^{-1}$ .

**Lemma 3.1.** *Let  $(\alpha, \beta, P, k)$  be given as in Theorem 1.6.*

- (i) *Let  $\chi$  be a 1-pure  $n$ -braid. If Theorem 1.6 is true for  $(\chi\alpha\chi^{-1}, \chi\beta\chi^{-1}, \pi_\chi(P), k)$ , then it is also true for  $(\alpha, \beta, P, k)$ .*
- (ii) *Let  $\chi$  be a  $P$ -straight, 1-unlinked  $n$ -braid with  $\chi\beta^k = \beta^k\chi$ . If Theorem 1.6 is true for  $(\alpha, \chi\beta\chi^{-1}, P, k)$ , then it is also true for  $(\alpha, \beta, P, k)$ .*

*Proof.* (i) Note that  $(\chi\alpha\chi^{-1})^k = (\chi\beta\chi^{-1})^k$ , that both  $\chi\alpha\chi^{-1}$  and  $\chi\beta\chi^{-1}$  are  $\pi_\chi(P)$ -pure by Lemma 2.10 (i), and that  $1 \in \pi_\chi(P)$  because  $1 \in P$  and  $\chi$  is 1-pure. Suppose Theorem 1.6 is true for  $(\chi\alpha\chi^{-1}, \chi\beta\chi^{-1}, \pi_\chi(P), k)$ , that is, there exists a  $\pi_\chi(P)$ -straight, 1-unlinked  $n$ -braid  $\gamma_1$  with  $\chi\beta\chi^{-1} = \gamma_1(\chi\alpha\chi^{-1})\gamma_1^{-1}$ . Let  $\gamma = \chi^{-1}\gamma_1\chi$ , then  $\beta = \gamma\alpha\gamma^{-1}$ . Since  $\gamma_1$  is  $\pi_\chi(P)$ -straight,  $\gamma$  is  $P$ -straight by Lemma 2.10 (ii). Since  $\gamma_1$  is 1-unlinked and  $\chi$  is 1-pure,  $\gamma$  is 1-unlinked by Lemma 2.10 (iii).

(ii) Since  $\chi$  commutes with  $\beta^k$ ,  $(\chi\beta\chi^{-1})^k = \chi\beta^k\chi^{-1} = \beta^k = \alpha^k$ . Because both  $\beta$  and  $\chi$  are  $P$ -pure,  $\chi\beta\chi^{-1}$  is  $P$ -pure by Lemma 2.10 (iv). Suppose Theorem 1.6 is true for  $(\alpha, \chi\beta\chi^{-1}, P, k)$ , that is, there exists a  $P$ -straight, 1-unlinked  $n$ -braid  $\gamma_1$  such that  $\chi\beta\chi^{-1} = \gamma_1\alpha\gamma_1^{-1}$ . Let  $\gamma = \chi^{-1}\gamma_1$ , then  $\beta = \gamma\alpha\gamma^{-1}$ . Since both  $\gamma_1$  and  $\chi$  are  $P$ -straight and 1-unlinked,  $\gamma$  is  $P$ -straight and 1-unlinked by Lemma 2.10 (iv).  $\square$

**Lemma 3.2.** *Let  $(\alpha, \beta, P, k)$  be given as in Theorem 1.6. If  $\alpha$  is either pseudo-Anosov or periodic, then Theorem 1.6 is true for  $(\alpha, \beta, P, k)$ .*

*Proof.* If  $\alpha$  is pseudo-Anosov, then  $\alpha = \beta$  by Lemma 2.1. If  $\alpha$  is central, then  $\alpha = \beta$  because  $\beta$  is conjugate to  $\alpha$ . In these two cases, we can take the identity as the desired conjugating element  $\gamma$ .

Suppose that  $\alpha$  is periodic and non-central. Then both  $\alpha$  and  $\beta$  are conjugate to  $\epsilon^m$  for some  $m \not\equiv 0 \pmod{n-1}$  by Lemma 2.2 since they are 1-pure and non-central. By Lemma 2.11, there exist 1-unlinked  $n$ -braids  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1\alpha\gamma_1^{-1} = \epsilon^m = \gamma_2\beta\gamma_2^{-1}$ . Let  $\gamma = \gamma_2^{-1}\gamma_1$ , then  $\beta = \gamma\alpha\gamma^{-1}$ . Because both  $\gamma_1$  and  $\gamma_2$  are 1-unlinked, so is  $\gamma$ . Because the first strand is the only pure strand of  $\alpha$  and  $1 \in P$ , we have  $P = \{1\}$ . Therefore  $\gamma$  is  $P$ -straight.  $\square$

**Definition 3.3.** Let  $\alpha$  be an  $n$ -braid with  $\mathcal{R}_{\text{ext}}(\alpha)$  standard. Then there exists a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$  such that  $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$  and  $\alpha$  can be expressed as

$$\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \dots \oplus \alpha_r)_{\mathbf{n}}.$$

In this case, the tubular  $r$ -braid  $\alpha_0$  of  $\alpha$  is specially denoted by  $\alpha_{\text{ext}}$ .

Note that for non-periodic reducible braids  $\alpha$  and  $\beta$ , if  $\alpha^k = \beta^k$  for a nonzero integer  $k$ , then

$$\emptyset \neq \mathcal{R}_{\text{ext}}(\alpha) = \mathcal{R}_{\text{ext}}(\alpha^k) = \mathcal{R}_{\text{ext}}(\beta^k) = \mathcal{R}_{\text{ext}}(\beta).$$

**Lemma 3.4.** *Let  $(\alpha, \beta, P, k)$  be given as in Theorem 1.6. If  $\mathcal{R}_{\text{ext}}(\alpha)$  is standard and  $\alpha_{\text{ext}} = \beta_{\text{ext}}$ , then Theorem 1.6 is true for  $(\alpha, \beta, P, k)$ .*

*Proof.* We will show this lemma by induction on the braid index  $n$ . If  $n = 2$ , Theorem 1.6 is obvious because  $B_{2,1}$  is infinite cyclic generated by  $\sigma_1^2$ : if  $\alpha = \sigma_1^{2p}$  and  $\beta = \sigma_1^{2q}$ , then  $\alpha^k = \beta^k$  implies  $p = q$ , and hence  $\alpha = \beta$  and the identity is a conjugating element from  $\alpha$  to  $\beta$ .

Suppose that  $n > 2$  and that the theorem is true for braids with less than  $n$  strands. Let  $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$  for an  $r$ -composition  $\mathbf{n}$  of  $n$ . Let  $\alpha_0 = \alpha_{\text{ext}} \in B_r$ . Since  $\alpha$  is 1-pure, so is  $\alpha_0$ , that is,  $\pi_{\alpha_0}(1) = 1$ . Let  $\{z_2, z_3, \dots, z_m\}$  be the set of all points other than 1 each of which is fixed by  $\pi_{\alpha_0}$ .

**Claim.** Without loss of generality, we may assume that  $\{z_2, \dots, z_m\} = \{2, \dots, m\}$  (i.e.  $\pi_{\alpha_0}(i) = i$  for all  $1 \leq i \leq m$ ) and each of the other cycles of  $\pi_{\alpha_0}$  is of the form  $(i + r_i, \dots, i + 2, i + 1)$  for some  $i \geq m$  and  $r_i \geq 2$ .

*Proof of Claim.* Choose an  $r$ -permutation  $\theta$  such that  $\theta(1) = 1$ ,  $\theta(\{z_2, \dots, z_m\}) = \{2, \dots, m\}$  and each cycle (of length  $\geq 2$ ) of  $\theta\pi_{\alpha_0}\theta^{-1}$  is of the form  $(i + r_i, \dots, i + 2, i + 1)$ . Note that  $\theta\pi_{\alpha_0}\theta^{-1}$  fixes each point of  $\{1, \dots, m\}$ . Let  $\zeta_0$  be an  $r$ -braid whose induced permutation is  $\theta$ , and let  $\zeta = \langle \zeta_0 \rangle_{\mathbf{n}}$ . Since  $\zeta_0$  is 1-pure,  $\zeta$  is also 1-pure. Applying Lemma 3.1 (i) to  $\zeta$  and  $(\alpha, \beta, P, k)$ , it suffices to show that Theorem 1.6 is true for  $(\zeta\alpha\zeta^{-1}, \zeta\beta\zeta^{-1}, \pi_\zeta(P), k)$ . Note that  $\mathcal{R}_{\text{ext}}(\zeta\alpha\zeta^{-1}) = \zeta * \mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\zeta_0 * \mathbf{n}}$  is standard and that  $(\zeta\alpha\zeta^{-1})_{\text{ext}} = \zeta_0\alpha_{\text{ext}}\zeta_0^{-1} = \zeta_0\beta_{\text{ext}}\zeta_0^{-1} = (\zeta\beta\zeta^{-1})_{\text{ext}}$ .  $\square$

Using the above claim, we assume that  $\pi_{\alpha_0}(i) = i$  for all  $1 \leq i \leq m$  and each of the other cycles of  $\pi_{\alpha_0}$  is of the form  $(i + r_i, \dots, i + 2, i + 1)$  for some  $i \geq m$  and  $r_i \geq 2$ . Then  $\mathbf{n}$ ,  $\alpha$  and  $\beta$  are as follows:

$$\begin{aligned} \mathbf{n} &= (n_1, \dots, n_m, \underbrace{n_{m+1}, \dots, n_{m+1}}_{r_{m+1}}, \dots, \underbrace{n_s, \dots, n_s}_{r_s}), \\ \alpha &= \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \dots \oplus \alpha_m \oplus (\alpha_{m+1,1} \oplus \dots \oplus \alpha_{m+1,r_{m+1}}) \oplus \dots \oplus (\alpha_{s,1} \oplus \dots \oplus \alpha_{s,r_s}))_{\mathbf{n}}, \\ \beta &= \langle \alpha_0 \rangle_{\mathbf{n}} (\beta_1 \oplus \dots \oplus \beta_m \oplus (\beta_{m+1,1} \oplus \dots \oplus \beta_{m+1,r_{m+1}}) \oplus \dots \oplus (\beta_{s,1} \oplus \dots \oplus \beta_{s,r_s}))_{\mathbf{n}}. \end{aligned}$$

By raising the power  $k$  large enough, we may assume that the lengths  $r_i$  of the cycles of  $\pi_{\alpha_0}$  are all divisors of  $k$ . Let  $k = r_i p_i$  for  $m < i \leq s$ . Then

$$\begin{aligned} \alpha^k &= \langle \alpha_0^k \rangle_{\mathbf{n}} (\alpha_1^k \oplus \dots \oplus \alpha_m^k \oplus (\tilde{\alpha}_{m+1,1}^{p_{m+1}} \oplus \dots \oplus \tilde{\alpha}_{m+1,r_{m+1}}^{p_{m+1}}) \oplus \dots \oplus (\tilde{\alpha}_{s,1}^{p_s} \oplus \dots \oplus \tilde{\alpha}_{s,r_s}^{p_s}))_{\mathbf{n}}, \\ \beta^k &= \langle \alpha_0^k \rangle_{\mathbf{n}} (\beta_1^k \oplus \dots \oplus \beta_m^k \oplus (\tilde{\beta}_{m+1,1}^{p_{m+1}} \oplus \dots \oplus \tilde{\beta}_{m+1,r_{m+1}}^{p_{m+1}}) \oplus \dots \oplus (\tilde{\beta}_{s,1}^{p_s} \oplus \dots \oplus \tilde{\beta}_{s,r_s}^{p_s}))_{\mathbf{n}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\alpha}_{i,j} &= \alpha_{i,j-r_i+1} \alpha_{i,j-r_i+2} \dots \alpha_{i,j-1} \alpha_{i,j}, \\ \tilde{\beta}_{i,j} &= \beta_{i,j-r_i+1} \beta_{i,j-r_i+2} \dots \beta_{i,j-1} \beta_{i,j} \end{aligned}$$

for  $m < i \leq s$  and  $1 \leq j \leq r_i$ . Hereafter we regard the second index  $j$  of  $(i, j)$  as being taken modulo  $r_i$ . Since  $\alpha^k = \beta^k$ , one has

$$\begin{aligned} \alpha_i^k &= \beta_i^k & \text{for } 1 \leq i \leq m, \\ \tilde{\alpha}_{i,j}^{p_i} &= \tilde{\beta}_{i,j}^{p_i} & \text{for } m < i \leq s \text{ and } 1 \leq j \leq r_i. \end{aligned}$$

Recall that  $\alpha$  and  $\beta$  are  $P$ -pure, hence  $\alpha_i$  and  $\beta_i$  are  $P_{\mathbf{n},i}$ -pure for  $1 \leq i \leq m$  by Lemma 2.15. Recall also that the induced permutation of  $\alpha_0$  fixes no point  $i > m$ , hence  $P_{\mathbf{n},i} = \emptyset$  for  $i > m$ .

From now on, we will construct an  $n$ -braid  $\gamma$  such that  $\beta = \gamma\alpha\gamma^{-1}$ . It will be of the form

$$\gamma = (\gamma_1 \oplus \dots \oplus \gamma_m \oplus (\gamma_{m+1,1} \oplus \dots \oplus \gamma_{m+1,r_{m+1}}) \oplus \dots \oplus (\gamma_{s,1} \oplus \dots \oplus \gamma_{s,r_s}))_{\mathbf{n}},$$

where  $\gamma_1$  is 1-unlinked and  $P_{\mathbf{n},1}$ -straight, and  $\gamma_i$  is  $P_{\mathbf{n},i}$ -straight for  $2 \leq i \leq m$ . Then  $\gamma$  is 1-unlinked by Lemma 2.15 (iii) because  $\gamma_1$  is 1-unlinked. And  $\gamma$  is  $P$ -straight by Lemma 2.15 (ii) because  $\gamma_i$  is  $P_{\mathbf{n},i}$ -straight for  $1 \leq i \leq m$  and  $P_{\mathbf{n},i} = \emptyset$  for  $i > m$ .

Note that  $\alpha_1^k = \beta_1^k$  and that  $1 \in P_{\mathbf{n},1}$  because  $1 \in P$ . By the induction hypothesis on the braid index, there exists a  $P_{\mathbf{n},1}$ -straight, 1-unlinked  $n_1$ -braid  $\gamma_1$  with  $\beta_1 = \gamma_1\alpha_1\gamma_1^{-1}$ .

Let  $2 \leq i \leq m$ . Note that  $\alpha_i^k = \beta_i^k$ . If  $P_{\mathbf{n},i} = \emptyset$ , there is an  $n_i$ -braid  $\gamma_i$  such that  $\beta_i = \gamma_i\alpha_i\gamma_i^{-1}$  by [Gon03]. Suppose  $P_{\mathbf{n},i} \neq \emptyset$ . Then there is an  $n_i$ -braid  $\zeta_i$  with  $1 \in \pi_{\zeta_i}(P_{\mathbf{n},i})$ . Since  $\alpha_i$  and  $\beta_i$  are

$P_{\mathbf{n},i}$ -pure,  $\zeta_i \alpha_i \zeta_i^{-1}$  and  $\zeta_i \beta_i \zeta_i^{-1}$  are  $\pi_{\zeta_i}(P_{\mathbf{n},i})$ -pure  $n_i$ -braids with  $(\zeta_i \alpha_i \zeta_i^{-1})^k = (\zeta_i \beta_i \zeta_i^{-1})^k$ . By the induction hypothesis on the braid index, Theorem 1.6 is true for  $(\zeta_i \alpha_i \zeta_i^{-1}, \zeta_i \beta_i \zeta_i^{-1}, \pi_{\zeta_i}(P_{\mathbf{n},i}), k)$ , hence there exists a  $\pi_{\zeta_i}(P_{\mathbf{n},i})$ -straight  $n_i$ -braid  $\chi_i$  such that  $\zeta_i \beta_i \zeta_i^{-1} = \chi_i (\zeta_i \alpha_i \zeta_i^{-1}) \chi_i^{-1}$ . Let  $\gamma_i = \zeta_i^{-1} \chi_i \zeta_i$ . Then  $\gamma_i$  is a  $P_{\mathbf{n},i}$ -straight  $n_i$ -braid with  $\beta_i = \gamma_i \alpha_i \gamma_i^{-1}$ .

Recall that  $\tilde{\alpha}_{i,r_i}^{p_i} = \tilde{\beta}_{i,r_i}^{p_i}$  for all  $m < i \leq s$ , which implies that there are  $\zeta_i \in B_{n_i}$  with

$$\tilde{\beta}_{i,r_i} = \zeta_i \tilde{\alpha}_{i,r_i} \zeta_i^{-1}.$$

For  $m < i \leq s$  and  $1 \leq j \leq r_i$ , define  $\gamma_{i,j}$  by

$$\gamma_{i,j} = (\beta_{i,j}^{-1} \cdots \beta_{i,2}^{-1} \beta_{i,1}^{-1}) \zeta_i (\alpha_{i,1} \alpha_{i,2} \cdots \alpha_{i,j}).$$

Then, for  $m < i \leq s$  and  $1 < j \leq r_i$ ,

$$\begin{aligned} \gamma_{i,r_i} \alpha_{i,1} \gamma_{i,1}^{-1} &= (\beta_{i,r_i}^{-1} \cdots \beta_{i,1}^{-1} \zeta_i \alpha_{i,1} \cdots \alpha_{i,r_i}) \alpha_{i,1} (\alpha_{i,1}^{-1} \zeta_i^{-1} \beta_{i,1}) = \tilde{\beta}_{i,r_i}^{-1} \zeta_i \tilde{\alpha}_{i,r_i} \zeta_i^{-1} \beta_{i,1} = \beta_{i,1}, \\ \gamma_{i,j-1} \alpha_{i,j} \gamma_{i,j}^{-1} &= (\beta_{i,j-1}^{-1} \cdots \beta_{i,1}^{-1} \zeta_i \alpha_{i,1} \cdots \alpha_{i,j-1}) \alpha_{i,j} (\alpha_{i,j}^{-1} \cdots \alpha_{i,1}^{-1} \zeta_i^{-1} \beta_{i,1} \cdots \beta_{i,j}) = \beta_{i,j}. \end{aligned}$$

Therefore

$$\gamma_{i,j-1} \alpha_{i,j} \gamma_{i,j}^{-1} = \beta_{i,j} \quad \text{for } m < i \leq s \text{ and } 1 \leq j \leq r_i.$$

So far, we have constructed the desired  $P$ -straight and 1-unlinked  $n$ -braid

$$\gamma = (\gamma_1 \oplus \cdots \oplus \gamma_m \oplus (\gamma_{m+1,1} \oplus \cdots \oplus \gamma_{m+1,r_{m+1}}) \oplus \cdots \oplus (\gamma_{s,1} \oplus \cdots \oplus \gamma_{s,r_s}))_{\mathbf{n}}.$$

It remains to show  $\beta = \gamma \alpha \gamma^{-1}$ , which will be done by a direct computation. In the following,  $\bigoplus_{i=1}^{\ell} \chi_i$  means  $\chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_{\ell}$ .

$$\begin{aligned} \gamma \alpha &= (\gamma_1 \oplus \cdots \oplus \gamma_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \gamma_{i,j})_{\mathbf{n}} \cdot \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \cdots \oplus \alpha_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \alpha_{i,j})_{\mathbf{n}} \\ &= \langle \alpha_0 \rangle_{\mathbf{n}} \cdot (\gamma_1 \oplus \cdots \oplus \gamma_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \gamma_{i,j-1})_{\mathbf{n}} \cdot (\alpha_1 \oplus \cdots \oplus \alpha_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \alpha_{i,j})_{\mathbf{n}} \\ &= \langle \alpha_0 \rangle_{\mathbf{n}} (\gamma_1 \alpha_1 \oplus \cdots \oplus \gamma_m \alpha_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \gamma_{i,j-1} \alpha_{i,j})_{\mathbf{n}}, \\ \beta \gamma &= \langle \alpha_0 \rangle_{\mathbf{n}} (\beta_1 \oplus \cdots \oplus \beta_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \beta_{i,j})_{\mathbf{n}} \cdot (\gamma_1 \oplus \cdots \oplus \gamma_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \gamma_{i,j})_{\mathbf{n}} \\ &= \langle \alpha_0 \rangle_{\mathbf{n}} (\beta_1 \gamma_1 \oplus \cdots \oplus \beta_m \gamma_m \oplus \bigoplus_{i=m+1}^s \bigoplus_{j=1}^{r_i} \beta_{i,j} \gamma_{i,j})_{\mathbf{n}}. \end{aligned}$$

Because  $\gamma_i \alpha_i \gamma_i^{-1} = \beta_i$  for  $1 \leq i \leq m$  and  $\gamma_{i,j-1} \alpha_{i,j} \gamma_{i,j}^{-1} = \beta_{i,j}$  for  $m < i \leq s$  and  $1 \leq j \leq r_i$ , we have  $\gamma \alpha = \beta \gamma$ , and hence  $\gamma \alpha \gamma^{-1} = \beta$ .  $\square$

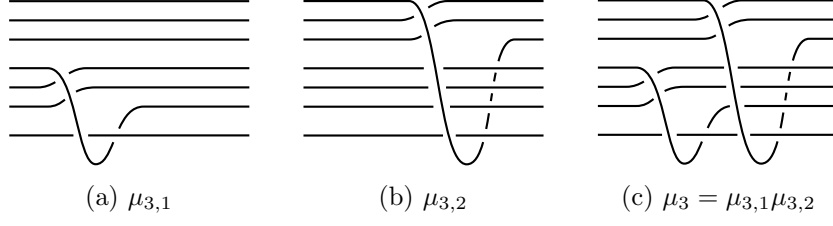
**Definition 3.5.** Let  $r$ ,  $s$  and  $d$  be integers with  $s \geq 2$ ,  $d \geq 1$  and  $r = ds + 1$ . For  $1 \leq j \leq d$ , define an  $r$ -braid  $\mu_{s,j}$  as

$$\mu_{s,j} = (\sigma_{js} \sigma_{j(s-1)} \cdots \sigma_2 \sigma_1) (\sigma_1 \sigma_2 \cdots \sigma_{(j-1)s} \sigma_{(j-1)(s+1)}).$$

Define  $\mu_s$  as  $\mu_s = \mu_{s,1} \mu_{s,2} \cdots \mu_{s,d}$ . See Figure 8 for the case  $r = 7$ ,  $s = 3$  and  $d = 2$ .

It is easy to see the following.

- $\mu_s$  is conjugate to  $\epsilon_{(r)}^d$  because  $(\mu_s)^s = \Delta_{(r)}^2 = (\epsilon_{(r)}^d)^s$ .
- For any  $1 \leq i, j \leq d$ ,  $\mu_{s,i}$  and  $\mu_{s,j}$  mutually commute.
- $\text{lk}(\mu_{s,j}) = 1$  for  $1 \leq j \leq d$ .

FIGURE 8.  $\mu_{3,1}$ ,  $\mu_{3,2}$  and  $\mu_3$  when  $r = 7$ 

**Lemma 3.6.** *Let  $P$  be a subset of  $\{1, \dots, n\}$  with  $1 \in P$ . Let  $\alpha$  be a  $P$ -pure  $n$ -braid with  $\mathcal{R}_{\text{ext}}(\alpha)$  standard, hence  $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$  for a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$ . Let  $\alpha_{\text{ext}}$  be periodic and non-central.*

- (i) *For each  $2 \leq i \leq r$ , there exists a  $P$ -straight  $n$ -braid  $\gamma$  such that  $\gamma\alpha = \alpha\gamma$  and  $\text{lk}(\gamma) = n_i$ .*
- (ii) *Let  $\chi = \langle \chi_0 \rangle_{\mathbf{n}} (\chi_1 \oplus \dots \oplus \chi_r)_{\mathbf{n}}$  be  $P$ -straight such that  $\chi_1$  is 1-unlinked. Then there exists a  $P$ -straight  $n$ -braid  $\gamma$  such that  $\gamma\alpha = \alpha\gamma$  and  $\text{lk}(\gamma) = -\text{lk}(\chi)$ .*

*Proof.* (i) Note that  $\alpha_{\text{ext}}$  is 1-pure because  $\alpha$  is 1-pure. In addition,  $\alpha_{\text{ext}}$  is periodic and non-central. Thus  $\alpha_{\text{ext}}$  is conjugate to  $\epsilon_{(r)}^m$  for some  $m \not\equiv 0 \pmod{r-1}$ . Let  $d = \gcd(m, r-1)$ ,  $m = dt$  and  $r-1 = ds$ .

**Claim.** Without loss of generality, we may assume  $\alpha_{\text{ext}} = \mu_s^t$ .

*Proof of Claim.* Assume that (i) holds for braids  $\alpha'$  with  $\alpha'_{\text{ext}} = \mu_s^t$ . Since  $\alpha_{\text{ext}}$  is conjugate to  $\epsilon_{(r)}^m = \epsilon_{(r)}^{dt}$  and  $\epsilon_{(r)}^d$  is conjugate to  $\mu_s$ ,  $\mu_s^t$  is conjugate to  $\alpha_{\text{ext}}$ . Since both  $\alpha_{\text{ext}}$  and  $\mu_s^t$  are 1-pure braids that are periodic and non-central, they have the first strand as the only pure strand. Thus there exists a 1-pure  $r$ -braid  $\zeta_0$  such that  $\mu_s^t = \zeta_0 \alpha_{\text{ext}} \zeta_0^{-1}$ . Let

$$\zeta = \langle \zeta_0 \rangle_{\mathbf{n}} \quad \text{and} \quad \beta = \zeta \alpha \zeta^{-1}.$$

Since  $\alpha$  is  $P$ -pure,  $\beta$  is  $\pi_{\zeta}(P)$ -pure. Since  $\zeta$  is 1-pure and  $1 \in P$ , we have  $1 \in \pi_{\zeta}(P)$ . Since  $\mathcal{R}_{\text{ext}}(\beta) = \zeta * \mathcal{R}_{\text{ext}}(\alpha) = \zeta * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\zeta_0 * \mathbf{n}}$ ,  $\mathcal{R}_{\text{ext}}(\beta)$  is standard and  $\beta_{\text{ext}} = \zeta_0 \alpha_{\text{ext}} \zeta_0^{-1} = \mu_s^t$ . Fix any  $2 \leq i \leq r$ . Since  $\zeta_0 * \mathbf{n} = (n_1, n'_2, \dots, n'_r)$ , where  $(n'_2, \dots, n'_r)$  is a rearrangement of  $(n_2, \dots, n_r)$ , there exists  $2 \leq j \leq r$  such that  $n_i = n'_j$ . By the assumption, there exists a  $\pi_{\zeta}(P)$ -straight  $n$ -braid  $\chi$  such that  $\chi\beta\chi^{-1} = \alpha$  and  $\text{lk}(\chi) = n'_j$ . Let  $\gamma = \zeta^{-1}\chi\zeta$ . Then  $\gamma\alpha\gamma^{-1} = \alpha$ . Because  $\gamma$  is  $P$ -straight with  $\text{lk}(\gamma) = \text{lk}(\chi) = n'_j = n_i$ , we are done.  $\square$

Now, we assume  $\alpha_{\text{ext}} = \mu_s^t$ . Then  $\alpha$  can be expressed as

$$\alpha = \langle \mu_s^t \rangle_{\mathbf{n}} (\alpha_1 \oplus (\alpha_{1,1} \oplus \alpha_{1,2} \oplus \dots \oplus \alpha_{1,s}) \oplus \dots \oplus (\alpha_{d,1} \oplus \alpha_{d,2} \oplus \dots \oplus \alpha_{d,s}))_{\mathbf{n}}.$$

For convenience, let  $[k, \ell]$  denote the integer  $(k-1)s + \ell + 1$  for  $1 \leq k \leq d$  and  $1 \leq \ell \leq s$ . Then

$$\mathbf{n} = (n_1, n_2, \dots, n_r) = (n_1, \underbrace{n_{[1,1]}, n_{[1,2]}, \dots, n_{[1,s]}}_s, \dots, \underbrace{n_{[d,1]}, n_{[d,2]}, \dots, n_{[d,s]}}_s).$$

Hereafter we regard the second index  $\ell$  of  $[k, \ell]$  as being taken modulo  $s$ . Notice the following.

- The induced permutation of  $\mu_s^t$  fixes 1 and maps  $[k, \ell]$  to  $[k, \ell - t]$ . Because  $\gcd(s, t) = 1$ , the induced permutation of  $\mu_s^t$  has a single fixed point and each of the other cycles has length  $s$ . Therefore

$$P = P_{\mathbf{n},1} \quad \text{and} \quad n_{[k,1]} = n_{[k,2]} = \dots = n_{[k,s]} \quad \text{for } 1 \leq k \leq d.$$

- The induced permutation of  $\mu_{s,j}$  maps  $[j, \ell]$  to  $[j, \ell - 1]$  for  $1 \leq \ell \leq s$ , and it fixes the other points.
- $\text{lk}_{[j,1]}(\mu_{s,j}) = 1$ , and  $\text{lk}_{[k,\ell]}(\mu_{s,j}) = 0$  if  $(k, \ell) \neq (j, 1)$ .

Because  $\gcd(s, t) = 1$ , there exist integers  $a > 0$  and  $b$  such that  $at + bs = 1$ . Fix any  $2 \leq i \leq r$ . Then  $n_i$  is equal to  $n_{[j,1]}$  for some  $1 \leq j \leq d$ . Define an  $n$ -braid  $\gamma$  to be

$$\gamma = \langle \mu_{s,j} \rangle_{\mathbf{n}} (\gamma_1 \oplus (\gamma_{1,1} \oplus \gamma_{1,2} \oplus \cdots \oplus \gamma_{1,s}) \oplus \cdots \oplus (\gamma_{d,1} \oplus \gamma_{d,2} \oplus \cdots \oplus \gamma_{d,s}))_{\mathbf{n}},$$

where  $\gamma_1 = 1$ ,  $\gamma_{k,\ell} = 1$  for  $k \neq j$  and

$$\gamma_{j,\ell} = \alpha_{j,\ell-(a-1)t} \alpha_{j,\ell-(a-2)t} \cdots \alpha_{j,\ell-2t} \alpha_{j,\ell-t} \alpha_{j,\ell} \quad \text{for } 1 \leq \ell \leq s.$$

Then  $\gamma$  is  $P$ -straight because  $P = P_{\mathbf{n},1}$ ,  $\mu_{s,j}$  is 1-pure, and  $\gamma_1 = 1$ . In addition, by Lemma 2.13,

$$\text{lk}(\gamma) = \text{lk}(\gamma_1) + \sum_{k=1}^d \sum_{\ell=1}^s n_{[k,\ell]} \text{lk}_{[k,\ell]}(\mu_{s,j}) = n_{[j,1]} \text{lk}_{[j,1]}(\mu_{s,j}) = n_{[j,1]} = n_i.$$

Now, it remains to show  $\alpha\gamma = \gamma\alpha$ . We will do it by a straightforward computation together with the following claim.

**Claim.** For  $1 \leq \ell \leq s$ , we have  $\alpha_{j,\ell-1} \gamma_{j,\ell} = \gamma_{j,\ell-t} \alpha_{j,\ell}$ .

*Proof of Claim.* Recall that  $\gamma_{j,\ell} = \alpha_{j,\ell-(a-1)t} \alpha_{j,\ell-(a-2)t} \cdots \alpha_{j,\ell-2t} \alpha_{j,\ell-t} \alpha_{j,\ell}$ . Hence

$$\begin{aligned} \alpha_{j,\ell-1} \gamma_{j,\ell} &= \alpha_{j,\ell-1} \alpha_{j,\ell-(a-1)t} \alpha_{j,\ell-(a-2)t} \cdots \alpha_{j,\ell-2t} \alpha_{j,\ell-t} \alpha_{j,\ell}, \\ \gamma_{j,\ell-t} \alpha_{j,\ell} &= \alpha_{j,\ell-at} \alpha_{j,\ell-(a-1)t} \alpha_{j,\ell-(a-2)t} \cdots \alpha_{j,\ell-2t} \alpha_{j,\ell-t} \alpha_{j,\ell}. \end{aligned}$$

Notice that  $\alpha_{j,\ell-1} = \alpha_{j,\ell-at}$  because  $at \equiv 1 \pmod{s}$ . Therefore  $\alpha_{j,\ell-1} \gamma_{j,\ell} = \gamma_{j,\ell-t} \alpha_{j,\ell}$ .  $\square$

For simplicity of notations, let

$$\begin{aligned} \tilde{\alpha}_k &= (\alpha_{k,1} \oplus \cdots \oplus \alpha_{k,s})_{\mathbf{n}_k}, & \tilde{\alpha}_k^{(p)} &= (\alpha_{k,p+1} \oplus \cdots \oplus \alpha_{k,p+s})_{\mathbf{n}_k}, \\ \tilde{\gamma}_k &= (\gamma_{k,1} \oplus \cdots \oplus \gamma_{k,s})_{\mathbf{n}_k}, & \tilde{\gamma}_k^{(p)} &= (\gamma_{k,p+1} \oplus \cdots \oplus \gamma_{k,p+s})_{\mathbf{n}_k}, \end{aligned}$$

where  $1 \leq k \leq d$ ,  $\mathbf{n}_k = (n_{[k,1]}, \dots, n_{[k,s]})$  and  $p$  is an integer. Then

$$\alpha = \langle \mu_s^t \rangle_{\mathbf{n}} (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}} \quad \text{and} \quad \gamma = \langle \mu_{s,j} \rangle_{\mathbf{n}} (\gamma_1 \oplus \tilde{\gamma}_1 \oplus \cdots \oplus \tilde{\gamma}_d)_{\mathbf{n}}.$$

Because  $\gamma_1 = 1$  and  $\tilde{\gamma}_k = 1$  for  $k \neq j$ , we just write  $\gamma = \langle \mu_{s,j} \rangle_{\mathbf{n}} (\cdots \oplus 1 \oplus \tilde{\gamma}_j \oplus 1 \oplus \cdots)_{\mathbf{n}}$ . Then

$$\begin{aligned} \alpha\gamma &= \langle \mu_s^t \rangle_{\mathbf{n}} (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}} \cdot \langle \mu_{s,j} \rangle_{\mathbf{n}} (\cdots \oplus 1 \oplus \tilde{\gamma}_j \oplus 1 \oplus \cdots)_{\mathbf{n}} \\ &= \langle \mu_s^t \rangle_{\mathbf{n}} \cdot \langle \mu_{s,j} \rangle_{\mathbf{n}} \cdot (\alpha_1 \oplus \cdots \oplus \tilde{\alpha}_{j-1} \oplus \tilde{\alpha}_j^{(-1)} \oplus \tilde{\alpha}_{j+1} \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}} \cdot (\cdots \oplus 1 \oplus \tilde{\gamma}_j \oplus 1 \oplus \cdots)_{\mathbf{n}} \\ &= \langle \mu_s^t \mu_{s,j} \rangle_{\mathbf{n}} (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_{j-1} \oplus \tilde{\alpha}_j^{(-1)} \tilde{\gamma}_j \oplus \tilde{\alpha}_{j+1} \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}}, \\ \gamma\alpha &= \langle \mu_{s,j} \rangle_{\mathbf{n}} (\cdots \oplus 1 \oplus \tilde{\gamma}_j \oplus 1 \oplus \cdots)_{\mathbf{n}} \cdot \langle \mu_s^t \rangle_{\mathbf{n}} (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}} \\ &= \langle \mu_{s,j} \rangle_{\mathbf{n}} \cdot \langle \mu_s^t \rangle_{\mathbf{n}} \cdot (\cdots \oplus 1 \oplus \tilde{\gamma}_j^{(-t)} \oplus 1 \oplus \cdots)_{\mathbf{n}} \cdot (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}} \\ &= \langle \mu_{s,j} \mu_s^t \rangle_{\mathbf{n}} (\alpha_1 \oplus \tilde{\alpha}_1 \oplus \cdots \oplus \tilde{\alpha}_{j-1} \oplus \tilde{\gamma}_j^{(-t)} \tilde{\alpha}_j \oplus \tilde{\alpha}_{j+1} \oplus \cdots \oplus \tilde{\alpha}_d)_{\mathbf{n}}. \end{aligned}$$

From the above equations, since  $\mu_{s,j} \mu_s = \mu_s \mu_{s,j}$ , we can see that  $\alpha\gamma = \gamma\alpha$  if and only if  $\tilde{\alpha}_j^{(-1)} \tilde{\gamma}_j = \tilde{\gamma}_j^{(-t)} \tilde{\alpha}_j$ . On the other hand,

$$\begin{aligned} \tilde{\alpha}_j^{(-1)} \tilde{\gamma}_j &= (\alpha_{j,s} \oplus \alpha_{j,1} \oplus \cdots \oplus \alpha_{j,s-1})_{\mathbf{n}_j} \cdot (\gamma_{j,1} \oplus \cdots \oplus \gamma_{j,s})_{\mathbf{n}_j} = \bigoplus_{\ell=1}^s \alpha_{j,\ell-1} \gamma_{j,\ell}, \\ \tilde{\gamma}_j^{(-t)} \tilde{\alpha}_j &= (\gamma_{j,1-t} \oplus \cdots \oplus \gamma_{j,s-t})_{\mathbf{n}_j} \cdot (\alpha_{j,1} \oplus \cdots \oplus \alpha_{j,s})_{\mathbf{n}_j} = \bigoplus_{\ell=1}^s \gamma_{j,\ell-t} \alpha_{j,\ell}, \end{aligned}$$

where  $\mathbf{n}_j = (n_{[j,1]}, n_{[j,2]}, \dots, n_{[j,s]})$ . By the above claim, we are done.

(ii) As  $\text{lk}(\chi_1) = 0$ , we have  $\text{lk}(\chi) = \sum_{i=2}^r n_i \text{lk}_i(\chi_0)$ . By (i), for each  $2 \leq i \leq r$ , there exists a  $P$ -straight  $n$ -braid  $\zeta_i$  such that  $\text{lk}(\zeta_i) = n_i$  and  $\zeta_i$  commutes with  $\alpha$ . Let  $\gamma_i = \zeta_i^{-\text{lk}_i(\chi_0)}$ , then  $\gamma_i$  is a  $P$ -straight  $n$ -braid such that it commutes with  $\alpha$  and  $\text{lk}(\gamma_i) = -n_i \text{lk}_i(\chi_0)$ . Let  $\gamma = \gamma_2 \gamma_3 \cdots \gamma_r$ . Then  $\gamma$  is a  $P$ -straight  $n$ -braid such that  $\gamma\alpha = \alpha\gamma$ . Moreover,  $\text{lk}(\gamma) = -\sum_{i=2}^r n_i \text{lk}_i(\chi_0) = -\text{lk}(\chi)$ .  $\square$

Now we are ready to prove Theorem 1.6.

**Proof of Theorem 1.6.** We will show the theorem by induction on the braid index  $n$ . If  $n = 2$ , Theorem 1.6 is obvious as we have observed in the proof of Lemma 3.4. Suppose that  $n > 2$  and that the theorem is true for braids with less than  $n$  strands.

Recall that  $1 \in P \subset \{1, \dots, n\}$ , and that  $\alpha$  and  $\beta$  are  $P$ -pure  $n$ -braids such that  $\alpha^k = \beta^k$  for some  $k \neq 0$ . If  $\alpha$  is either pseudo-Anosov or periodic, the theorem is true by Lemma 3.2. Thus we assume that  $\alpha$  is reducible and non-periodic.

If  $\mathcal{R}_{\text{ext}}(\alpha)$  is not standard, choose  $\zeta \in B_{n,1}$  such that  $\zeta * \mathcal{R}_{\text{ext}}(\alpha) = \mathcal{R}_{\text{ext}}(\zeta\alpha\zeta^{-1})$  is standard. By Lemma 3.1, it suffices to prove the theorem for  $(\zeta\alpha\zeta^{-1}, \zeta\beta\zeta^{-1}, \pi_\zeta(P), k)$ . Therefore, without loss of generality, we assume that  $\mathcal{R}_{\text{ext}}(\alpha)$  is standard.

There exists a composition  $\mathbf{n} = (n_1, \dots, n_r)$  of  $n$  such that  $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{R}_{\text{ext}}(\beta) = \mathcal{C}_{\mathbf{n}}$ , hence  $\alpha$  and  $\beta$  are expressed as

$$\alpha = \langle \alpha_0 \rangle_{\mathbf{n}} (\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r)_{\mathbf{n}} \quad \text{and} \quad \beta = \langle \beta_0 \rangle_{\mathbf{n}} (\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_r)_{\mathbf{n}}.$$

Since  $\alpha$  and  $\beta$  are  $P$ -pure,  $\alpha_i$  and  $\beta_i$  are  $P_{\mathbf{n},i}$ -pure for  $i = 0, 1$  by Lemma 2.15. In particular, because  $\alpha$  and  $\beta$  are 1-pure, the braids  $\alpha_0$  and  $\beta_0$  are 1-pure. Hence

$$\alpha^k = \langle \alpha_0^k \rangle_{\mathbf{n}} (\alpha_1^k \oplus \cdots)_{\mathbf{n}} \quad \text{and} \quad \beta^k = \langle \beta_0^k \rangle_{\mathbf{n}} (\beta_1^k \oplus \cdots)_{\mathbf{n}}.$$

(Here the second interior braid of  $\alpha^k$  is not necessarily  $\alpha_2^k$  unlike the first interior braid  $\alpha_1^k$ .) Since  $\alpha^k = \beta^k$ , we have  $\alpha_0^k = \beta_0^k$  and  $\alpha_1^k = \beta_1^k$ .

Note that  $\alpha_0$  is periodic or pseudo-Anosov. If  $\alpha_0$  is central, then it is obvious that  $\alpha_0 = \beta_0$ . If  $\alpha_0$  is pseudo-Anosov, then  $\alpha_0 = \beta_0$  by Lemma 2.1. For these two cases, we are done by Lemma 3.4. Therefore we assume that  $\alpha_0$  is periodic and non-central.

Since  $\alpha^k = \beta^k$ , there exists  $\zeta \in B_n$  with  $\beta = \zeta\alpha\zeta^{-1}$  by Theorem 1.2. Notice that

$$\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{R}_{\text{ext}}(\alpha^k) = \mathcal{R}_{\text{ext}}(\beta^k) = \mathcal{R}_{\text{ext}}(\beta) = \mathcal{R}_{\text{ext}}(\zeta\alpha\zeta^{-1}) = \zeta * \mathcal{R}_{\text{ext}}(\alpha),$$

i.e.  $\zeta$  preserves the curve system  $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$ . Hence  $\zeta$  can be expressed as

$$\zeta = \langle \zeta_0 \rangle_{\mathbf{n}} (\zeta_1 \oplus \zeta_2 \oplus \cdots \oplus \zeta_r)_{\mathbf{n}}.$$

We will replace  $\zeta_1$  in the above expression of  $\zeta$  with another braid  $\xi_1$  in order to make it  $P$ -straight, and then will multiply it by another  $n$ -braid  $\xi'$  in order to make it 1-unlinked.

Because  $\alpha_0$  and  $\beta_0$  are 1-pure, periodic and non-central, they have the first strand as the only pure strand by Corollary 2.3. Hence the  $r$ -braid  $\zeta_0$  must be 1-pure because  $\beta_0 = \zeta_0\alpha_0\zeta_0^{-1}$ . In addition,  $P_{\mathbf{n},0} = \{1\}$  and  $P = P_{\mathbf{n},1}$ . Recall that  $\alpha_1$  and  $\beta_1$  are  $P_{\mathbf{n},1}$ -pure. Because  $\alpha_1^k = \beta_1^k$  and  $1 \in P_{\mathbf{n},1}$ , there exists a  $P_{\mathbf{n},1}$ -straight, 1-unlinked  $n_1$ -braid  $\xi_1$  such that  $\beta_1 = \xi_1\alpha_1\xi_1^{-1}$ , by the induction hypothesis on the braid index. Let

$$\xi = \langle \zeta_0 \rangle_{\mathbf{n}} (\xi_1 \oplus \zeta_2 \oplus \zeta_3 \oplus \cdots \oplus \zeta_r)_{\mathbf{n}}.$$

Then  $\xi$  is  $P$ -straight since  $P = P_{n,1}$ ,  $\zeta_0$  is 1-pure and  $\xi_1$  is  $P_{n,1}$ -straight. Notice that  $\zeta$  and  $\xi$  are the same except for the first interior braids,  $\zeta_1$  and  $\xi_1$ . Notice also that  $\zeta_1\alpha_1\zeta_1^{-1} = \beta_1 = \xi_1\alpha_1\xi_1^{-1}$ . Therefore  $\xi\alpha\xi^{-1} = \zeta\alpha\zeta^{-1} = \beta$ .

By Lemma 3.6 (ii), there exists a  $P$ -straight  $n$ -braid  $\xi'$  such that  $\xi'\alpha = \alpha\xi'$  and  $\text{lk}(\xi') = -\text{lk}(\xi)$ . Let  $\gamma = \xi\xi'$ . Then  $\gamma$  is  $P$ -straight and 1-unlinked, and  $\gamma\alpha\gamma^{-1} = \xi\alpha\xi^{-1} = \beta$ .  $\square$

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