THE EXACT SOLUTIONS
FOR TWO CHANNELS DISSIPATION MODEL

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Abstract

The exact solutions of the linear two channels dissipation model in different way are presented. Firstly we analyze the steady state, traveling wave and particular solutions of this model. Then by using elimination method and canonical transformation the model becomes telegraph or Klein Gordon equation in the second order partial differential equation. In the Klein Gordon equation we get particular solution, so we also obtain particular solution of linear two channel dissipation model. In the form telegraph equation we solve by Fourier transform and obtain the solution of linear two channels dissipation model in the Fourier representation formula. Finally we find the exact solution by characteristic method in the integral form of the linear two channel dissipation model.

1. Introduction

Many process of fundamental equation occurring in the natural and physical sciences are obtained from conservation laws. Conservation laws are just balance laws, equations expressing the fact that some quantity is balanced throughout process. Mathematically, conservation laws usually translate into differential equations. In this paper let us consider *u(x; t)* and *v(x; t)* that depend on single spatial variable x and time $t > 0$. We assume that u and v are densities or concentrations measured in amount per unit volume that flow with speeds *c1(x)* and *c2(x)* respectively. Two groups of quantity have interactions, and then van Beckum(2003) arrived at the following set of equations for the concentrations *u(x; t)* in the first group and *v(x; t)* in the second group:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} + c_1(x)\frac{\partial u(x,t)}{\partial x} &= \alpha\, v(x,t) - \alpha\, u(x,t) \\ \frac{\partial v(x,t)}{\partial t} + c_2(x)\frac{\partial v(x,t)}{\partial x} &= \alpha\, u(x,t) - \alpha\, v(x,t) \end{aligned} \qquad (1)$$

The constant α is the coefficient of exchanging concentrations. The approximation of the classical solution of the above equations that have boundary and initial value conditions has been investigated Sumardi(2005) by using finite difference method in case $c_1(x) = -c_2(x) = 1$.

Sumardi(2007) also wrote approximation solution by Immersed Interface Method for case $c_1(x), c_2(x)$ discontinue constants. In this paper the exact solutions of the linear two channel dissipation model which have constant speed: $c_1(x) = c_1$ and $c_2(x) = c_2$ are presented. By using elimination method and canonical transformation the model becomes telegraph equation in second order partial differential equation. The particular solutions of Klein Gordon equation has been investigated by Polyanin(2002). The initial value problem of telegraph equation has been solved by Fourier transforms method Evans(1998) and Pinsky(1998).

2. Steady State Solution and traveling wave solution for two channels dissipation model

Steady state solutions for two channels dissipation model are solutions that are independent in time. Thus the steady state solutions of the equation (1), which have constant speed: $c_1(x) = c_1$ and $c_2(x) = c_2$ are solutions of the system ordinary differential equations:

$$\begin{aligned} c_1 \frac{du(x)}{dx} &= \alpha\, v(x) - \alpha\, u(x) \\ c_2 \frac{dv(x)}{dx} &= \alpha\, u(x) - \alpha\, v(x) \end{aligned} \quad . \qquad (2)$$

It is not difficult to solve the system by some method in theory of ordinary differential equations. The solution in case $c_1 \neq c_2$:

$$\begin{aligned} u(x) &= A + Bc_2 e^{\frac{-\alpha(c_1+c_2)x}{c_1c_2}} \\ v(x) &= A - Bc_1 e^{\frac{-\alpha(c_1+c_2)x}{c_1c_2}} \end{aligned} \qquad (3.a)$$

and in case $c_1 + c_2 = 0$:

$$\begin{aligned} U(x) &= Ax + B \\ V(x) &= Ax + B + \frac{Ac_1}{\alpha} \end{aligned} \qquad (3.b)$$

with constants A and B in equation (3.a) and (3.b) that are determined by boundary conditions *u(x)* and *v(x)* for finite real number $x \in \Re$. We also have if we solve the time-dependent system taking as initial functions the steady state, then these functions are the solution.

We look for constants traveling wave solutions with speed *s* by setting

$$u(x,t) = U(z), \quad v(x,t) = V(z), \; z = \alpha(x - st)$$

Two channels dissipation model reduce in ordinary differential system:

$$\begin{aligned} (-s + c_1)\frac{dU}{dz} &= V - U \\ (-s + c_2)\frac{dU}{dz} &= U - V \end{aligned} \qquad (4)$$

that have the same form with the equation (2), the solution in case $s \neq \frac{1}{2}(c_1 + c_2)$ with $s \neq c_1$ and $s \neq c_2$:

$$U(z) = A + B(-s + c_2)e^{\frac{-\alpha(c_1+c_2-2s)(x-st)}{s^2 - s(c_1+c_2)+c_1c_2}}$$
$$V(z) = A - B(-s + c_1)e^{\frac{-\alpha(c_1+c_2-2s)(x-st)}{s^2 - s(c_1+c_2)+c_1c_2}} \quad (5.a)$$

and in case $s = \frac{1}{2}(c_1 + c_2)$:

$$U(x) = A(x - st) + B = A(x - \frac{t}{2}(c_1 + c_2)) + B$$
$$V(x) = A(x - st) + B + \frac{A(-s + c_1)}{\alpha} = A(x - \frac{t}{2}(c_1 + c_2) + \frac{c_1 - c_2}{2\alpha}) + B \quad . \quad (5.b)$$

By compare the steady solution and the traveling wave solution for two channels dissipation model, we see that the steady state solution is traveling wave solution with speed zero. Both of the steady and traveling wave solutions are exponential and linear function that depending to speed c_1 and c_2. Parameter *s* in (5.a) will lead to speed of traveling wave which is travels forever, with constant speed and undisturbed shape. Constant *A* and *B* are arbitrary.

3. Particular Solutions for two channels dissipation model

Firstly we find the particular solutions for two channel dissipation model by Anzatz method. Suppose the particular solution in the form

$$u(x,t) = Ae^{x+mt}$$
$$v(x,t) = Be^{x+mt} \quad . \quad (6)$$

Then we find *A*, *B* and *m* in order to satisfy equation (1), we obtain system equation

$$(m + c_1 + \alpha)A - \alpha B = 0$$
$$-\alpha A + (m + c_2 + \alpha)B = 0 \quad . \quad (7)$$

To get non trivial solution, we have

$$\begin{vmatrix} m + c_1 + \alpha & -\alpha \\ -\alpha & m + c_2 + \alpha \end{vmatrix} = 0 \quad (8)$$

so we obtain the quadratic equation

$$m^2 + (c_1 + c_2 + \alpha)m + c_1c_2 + \alpha(c_1 + c_2) = 0 \quad (9)$$

that has two distinct real square roots, since the discriminant of the quadratic equation is $D = (c_1 - c_2)^2 + 4\alpha^2 > 0$. Hence we obtain

$$m_{1,2} = \frac{-c_1 - c_2 - 2\alpha \pm \sqrt{D}}{2} . \quad (10)$$

Substitute (10) into (7), we obtain relationship *A* and *B*:

$$B = \frac{c_1 - c_2 \pm \sqrt{D}}{2\alpha} A \quad (11)$$

so we obtain particular solutions:

$$u(x,t) = Ae^{x+\frac{-c_1-c_2-2\alpha\pm\sqrt{D}}{2}t}$$
$$v(x,t) = \frac{c_1 - c_2 \pm \sqrt{D}}{2\alpha} Ae^{x+\frac{-c_1-c_2-2\alpha\pm\sqrt{D}}{2}t}. \qquad (12)$$

We also obtain particular solutions for two channels dissipation model by elimination method, so we get one second order hyperbolic partial differential equations. The equation is transformed in the Klein Gordon equations. We can see that the Klein Gordon equation has many particular equations from Polyanin(2002).

Let equations (1) be written in operator diferential:

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial u}{\partial x} + \alpha\right) u(x,t) - \alpha\, v(x,t) = 0$$
$$-\alpha\, u(x,t) + \left(\frac{\partial}{\partial t} + c_2 \frac{\partial u}{\partial x} + \alpha\right) v(x,t) = 0 \qquad (13)$$

If the first equation multiplies by $\frac{\partial}{\partial t} + c_2 \frac{\partial u}{\partial x} + \alpha$ and the second equation multiplies by α then we can eliminate *v(x,t)*, so it is obtained

$$\frac{\partial^2 u}{\partial t^2} + (c_1 + c_2)\frac{\partial^2 u}{\partial t \partial x} + c_1 c_2 \frac{\partial^2 u}{\partial x^2} + 2\alpha \frac{\partial u}{\partial t} + \alpha(c_1 + c_2)\frac{\partial u}{\partial x} = 0 \qquad (14)$$

In order to solve equation (14) consider the changes variables in case $c_1 \neq c_2$

$$\eta = x - \frac{1}{2}(c_1 + c_2)t$$
$$\tau = \frac{1}{2}(c_1 - c_2)t \qquad (15)$$

Using the Chain rule from Calculus for transforming partial derivatives for functions of two variables we have the Telegraph equation:

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{4\alpha}{c_1 - c_2}\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \eta^2} = 0 \qquad (16)$$

To put this in more convenient form, let

$$\beta = \frac{2\alpha}{c_1 - c_2}, \qquad (17)$$

resulting in the equation

$$\frac{\partial^2 u}{\partial \tau^2} + \beta \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \eta^2} = 0 \qquad (18)$$

The Telegraph equation is transformed in Klein Gordon equation by

$$u(\eta,\tau) = e^{-\beta\tau} w(\eta,\tau) \qquad (19)$$

so we obtain

$$\frac{\partial^2 w}{\partial \tau^2} - \frac{\partial^2 w}{\partial \eta^2} - \beta^2 w = 0 \qquad (20)$$

Based on the book by Polyanin (2002) we obtain the particular solutions of Klein Gordon equation (20):

$$w(\eta,\tau) = e^{\pm\beta\tau}(A\eta + B) \tag{21}$$

$$w(\eta,\tau) = \cos(\eta\sqrt{\beta^2+\mu^2})(A\cos(\mu\tau) + B\sin(\mu\tau)) \tag{22}$$

$$w(\eta,\tau) = \sin(\eta\sqrt{\beta^2+\mu^2})(A\cos(\mu\tau) + B\sin(\mu\tau)) \tag{23}$$

$$w(\eta,\tau) = e^{\pm\tau\sqrt{\beta^2-\mu^2}}(A\cos(\mu\eta) + B\sin(\mu\eta)) \tag{24}$$

$$w(\eta,\tau) = e^{\pm\tau\sqrt{\beta^2-\mu^2}}(Ae^{\mu\eta} + Be^{-\mu\eta}) \tag{25}$$

$$w(\eta,\tau) = AI_0(\xi) + BK_0(\xi), \quad \xi = \beta\sqrt{(\tau+C_1)^2 - (\eta+C_2)^2} \tag{26}$$

where A, B, C_1 and C_2 are arbitrary constants, $I_0(\xi)$ and $K_0(\xi)$ are the modified Bessel functions. Then we work our way backwards with substituting the inverse transformation (13), (15), (17) and (19) we obtain the result below.

From the equation (21), we have

$$\begin{aligned} u(x,t) &= A(x - \frac{t}{2}(c_1+c_2)) + B \\ v(x,t) &= A(x - \frac{t}{2}(c_1+c_2) + \frac{c_1-c_2}{2\alpha}) + B \end{aligned} \tag{27}$$

$$\begin{aligned} u(x,t) &= e^{-2\alpha t}\left(A(x - \frac{t}{2}(c_1+c_2)) + B\right) \\ v(x,t) &= -e^{-2\alpha t}\left(A(x - \frac{t}{2}(c_1+c_2) + \frac{c_1-c_2}{2\alpha}) + B\right) \end{aligned} \tag{28}$$

From the equation (22), we have the particular solution:

$$u(x,t) = e^{-\alpha t}\cos\left((x - \frac{(c_1+c_2)t}{2})\sqrt{\beta^2+\mu^2}\right)\left(\cos(\mu\frac{(c_1-c_2)t}{2})A + \sin(\mu\frac{(c_1-c_2)t}{2})B\right)$$

$$\begin{aligned} v(x,t) = {} & A\frac{(c_2-c_1)\mu}{2\alpha}e^{-\alpha t}\cos\left((x - \frac{(c_1+c_2)t}{2})\sqrt{\beta^2+\mu^2}\right)\sin(\mu\frac{(c_1-c_2)t}{2}) \\ & + A\left(\frac{c_1+c_2}{2\alpha}\sqrt{\beta^2+\mu^2} - \frac{1}{\alpha}\right)e^{-\alpha t}\sin\left((x - \frac{(c_1+c_2)t}{2})\sqrt{\beta^2+\mu^2}\right)\cos(\mu\frac{(c_1-c_2)t}{2}) \\ & + B\frac{(c_1-c_2)\mu}{2\alpha}e^{-\alpha t}\cos\left((x - \frac{(c_1+c_2)t}{2})\sqrt{\beta^2+\mu^2}\right)\cos(\mu\frac{(c_1-c_2)t}{2}) \\ & + B\left(\frac{c_1+c_2}{2\alpha}\sqrt{\beta^2+\mu^2} - \frac{1}{\alpha}\right)e^{-\alpha t}\sin\left((x - \frac{(c_1+c_2)t}{2})\sqrt{\beta^2+\mu^2}\right)\sin(\mu\frac{(c_1-c_2)t}{2}) \end{aligned} \tag{29}$$

We also get particular solutions from equation (23)-(26).

4. The solution for initial value of two channels dissipation model

For the last section we will investigate the initial value problem for two channels dissipation model. Let

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + c_1 \frac{\partial u(x,t)}{\partial x} = \alpha\, v(x,t) - \alpha\, u(x,t) \\ &\frac{\partial v(x,t)}{\partial t} + c_2 \frac{\partial v(x,t)}{\partial x} = \alpha\, u(x,t) - \alpha\, v(x,t) \end{aligned} \qquad (30.a)$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \qquad x \in R, t \geq 0 \qquad (30.b)$$

For special case we take $c_1 = c_2 = c$, so we have

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + c \frac{\partial u(x,t)}{\partial x} = \alpha\, v(x,t) - \alpha\, u(x,t) \\ &\frac{\partial v(x,t)}{\partial t} + c \frac{\partial v(x,t)}{\partial x} = \alpha\, u(x,t) - \alpha\, v(x,t) \end{aligned} \qquad (31)$$

Adding two equations in (31) then we have advection equation, which have solution:

$$u(x,t) + v(x,t) = F(x - ct) \qquad (32)$$

where $F(x-ct)$ arbitrary function. Substitute (32) in first equation (31)

$$\frac{\partial u(x,t)}{\partial t} + c \frac{\partial u(x,t)}{\partial x} = -2\alpha\, u(x,t) + \alpha F(x - ct) \qquad (33)$$

then solve the equation (31), hence we have general solution;

$$\begin{aligned} &u(x,t) = \frac{1}{2}\left(F(x-ct) + e^{-2\alpha t} G(x-ct)\right) \\ &v(x,t) = \frac{1}{2}\left(F(x-ct) - e^{-2\alpha t} G(x-ct)\right) \end{aligned} \qquad (34)$$

Applying the initial condition (30.b), Thus we have the solution initial value (30.a)-(30.b) in case $c_1 = c_2 = c$:

$$\begin{aligned} &u(x,t) = \frac{1}{2}\left(u_0(x-ct) + v_0(x-ct) + e^{-2\alpha t}(u_0(x-ct) - v_0(x-ct))\right) \\ &v(x,t) = \frac{1}{2}\left(u_0(x-ct) + v_0(x-ct) - e^{-2\alpha t}(u_0(x-ct) - v_0(x-ct))\right) \end{aligned} \qquad (35)$$

In case $c_1 \neq c_2$ using elimination method and transformation variable of equation (15), we get new initial value problem in telegraph equation:

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{4\alpha}{c_1 - c_2}\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \eta^2} = 0 \qquad (36.a)$$

$$\begin{aligned} &u(\eta,0) = f(\eta) = u_0(\eta) \\ &\left.\frac{\partial u}{\partial \tau}\right|_{\tau=0} = g(\eta) = \frac{2}{c_1 - c_2}\left(\alpha\, v_0(\eta) - \alpha\, u_0(\eta) - c_1 u_0'(\eta)\right) \end{aligned} \qquad (36.b)$$

To solve the initial value problem above, we look for $u(\eta,\tau)$ in terms of its Fourier transform

$$u(\eta,\tau) = \int_{-\infty}^{\infty} U(\mu,\tau) e^{i\eta\mu} d\mu \tag{37.a}$$

$$U(\mu,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\eta,\tau) e^{-i\eta\mu} d\eta \tag{37.b}$$

and formally apply the operations implied by (36.a)

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{4\alpha}{c_1 - c_2}\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \eta^2} = \int_{-\infty}^{\infty}\left(\frac{\partial^2 U}{\partial \tau^2} + \frac{4\alpha}{c_1 - c_2}\frac{\partial U}{\partial \tau} + \mu^2 U\right) e^{i\eta\mu} d\mu\,. \tag{38}$$

Therefore we solve the ordinary differential equation

$$\frac{\partial^2 U}{\partial \tau^2} + \frac{4\alpha}{c_1 - c_2}\frac{\partial U}{\partial \tau} + \mu^2 U = 0 \tag{39}$$

with initial conditions

$$U(\mu,\tau) = F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\eta) e^{-i\eta\mu} d\eta \tag{40.a}$$

$$\left.\frac{\partial U(\mu,\tau)}{\partial \mu}\right|_{\tau=0} = G(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\eta) e^{-i\eta\mu} d\eta \tag{40.b}$$

We seek a solution (39) of the form

$$U(\mu,\tau) = Be^{\gamma\tau}, \qquad B,\gamma \in C \tag{41}$$

Plugging into (39), we deduce $\gamma^2 + \frac{4\alpha}{c_1 - c_2}\gamma + \mu^2 = 0$; hence

$$\gamma = -\left(\frac{4\alpha}{c_1 - c_2}\right) \pm \sqrt{\left(\frac{4\alpha}{c_1 - c_2}\right)^2 - \mu^2}\,.$$

Consequently

$$U(\mu,\tau) = \begin{cases} e^{\left(\frac{4\alpha}{c_1 - c_2}\right)\tau}\left(B_1(\mu)e^{\gamma(\mu)\tau} + B_2(\mu)e^{-\gamma(\mu)\tau}\right) & \text{if } |\mu| \le \left|\frac{4\alpha}{c_1 - c_2}\right| \\ e^{\left(\frac{4\alpha}{c_1 - c_2}\right)\tau}\left(B_1(\mu)e^{i\delta(\mu)\tau} + B_2(\mu)e^{-i\delta(\mu)\tau}\right) & \text{if } |\mu| > \left|\frac{4\alpha}{c_1 - c_2}\right| \end{cases} \tag{42}$$

for $\gamma(\mu) = \sqrt{\left(\frac{2\alpha}{c_1 - c_2}\right)^2 - \mu^2}$, $\delta(\mu) = \sqrt{\mu^2 - \left(\frac{2\alpha}{c_1 - c_2}\right)^2}$ where $B_1(\mu)$ and $B_2(\mu)$ are selected so that

$$F(\mu) = B_1(\mu) + B_2(\mu)$$

and

$$G(\mu) = \begin{cases} B_1(\mu)\left(\gamma(\mu) - \left(\dfrac{2\alpha}{c_1 - c_2}\right)\right) + B_2(\mu)\left(-\gamma(\mu) - \left(\dfrac{2\alpha}{c_1 - c_2}\right)\right) & \text{if} \ |\mu| \le \left|\dfrac{2\alpha}{c_1 - c_2}\right| \\ B_1(\mu)\left(i\delta(\mu) - \left(\dfrac{2\alpha}{c_1 - c_2}\right)\right) + B_2(\mu)\left(-i\delta(\mu) - \left(\dfrac{2\alpha}{c_1 - c_2}\right)\right) & \text{if} \ |\mu| > \left|\dfrac{2\alpha}{c_1 - c_2}\right| \end{cases} \quad (43)$$

We thereby obtain the Fourier representation formula:

$$\begin{aligned} u(\eta,\tau) = & \int_{|\mu| \le \left|\frac{2\alpha}{c_1-c_2}\right|} e^{\left(\frac{2\alpha}{c_1-c_2}\right)\tau} \left(B_1(\mu)e^{\gamma(\mu)\tau} + B_2(\mu)e^{-\gamma(\mu)\tau}\right) e^{i\eta\mu}\, d\mu \\ & + \int_{|\mu| > \left|\frac{2\alpha}{c_1-c_2}\right|} e^{\left(\frac{2\alpha}{c_1-c_2}\right)\tau} \left(B_1(\mu)e^{i\delta(\mu)\tau} + B_2(\mu)e^{-i\delta(\mu)\tau}\right) e^{i\eta\mu}\, d\mu \end{aligned} \quad (44)$$

Then we work our way backwards with substituting the inverse transformation (15) we obtain the result:

$$\begin{aligned} u(x,t) = & \int_{|\mu| \le \left|\frac{2\alpha}{c_1-c_2}\right|} e^{-\alpha t} \left(B_1(\mu) e^{\frac{t}{2}\sqrt{4\alpha^2 - \mu^2 (c_1+c_2)^2}} + B_2(\mu) e^{-\frac{t}{2}\sqrt{4\alpha^2 - \mu^2 (c_1+c_2)^2}} \right) e^{i(x - \frac{t}{2}(c_1+c_2)\mu}\, d\mu \\ & + \int_{|\mu| > \left|\frac{2\alpha}{c_1-c_2}\right|} e^{\left(\frac{4\alpha}{c_1-c_2}\right)\tau} \left(B_1(\mu) e^{i\frac{t}{2}\sqrt{\mu^2 (c_1+c_2)^2 - 4\alpha^2}} + B_2(\mu) e^{-i\delta(\mu)\tau} \right) e^{i\frac{t}{2}\sqrt{\mu^2 (c_1+c_2)^2 - 4\alpha^2}}\, d\mu \end{aligned} \quad (45)$$

and

$$v(x,t) = \frac{1}{\alpha}\left(\frac{\partial u(x,t)}{\partial t} + c_1 \frac{\partial u(x,t)}{\partial x} + \alpha u(x,t) \right) \ . \quad (46)$$

The other method for solving initial value problem we will use the characteristic method. The fundamental idea with hyperbolic equations is the notion of the characteristics, curves in space time along which these signal are propagated. In the curves along the partial differential equations can be reduced to simple form for example, system of ordinary differential equations. The characteristic are the curves along which information is transmitted in the system.

Let $(x,t) \in \Re^2$ and every $s \in \Re$ that defined by

$$\begin{aligned} w(s) &= u(x + c_1 s, t + s) \\ z(s) &= v(x + c_2 s, t + s) \end{aligned} . \quad (47)$$

Furthermore we obtain

$$\frac{dw(s)}{ds} = u_t(x + c_1 s, t+s) + u_x(x + c_1 s, t+s) = \alpha\left(v(x + c_1 s, t+s) - u(x + c_1 s, t+s)\right) \quad (48)$$

$$\frac{dz(s)}{ds} = v_t(x + c_2 s, t+s) + v_x(x + c_2 s, t+s) = \alpha\left(u(x + c_2 s, t+s) - v(x + c_2 s, t+s)\right) \quad (49)$$

The equation (46) is integrated from $s = -t$ to $s = 0$, it is obtained

$$\int_{-t}^{0} \frac{dw(s)}{ds} ds = \int_{-t}^{0} \alpha\big(v(x + c_1 s, t + s) - u(x + c_1 s, t + s)\big) ds$$

$$w(0) - w(-s) = \int_{0}^{t} \alpha\big(v(x + c_1 (s - t), s) - u(x + c_1 (s - t), s)\big) ds \qquad (50)$$

and because of

$$w(0) - w(-t) = u(x,t) - u(x,0) = u(x,t) - u(x - c_1 t, 0) = u(x,t) - u_0 (x - c_1 t) \qquad (51)$$

so we have the solution:

$$u(x,t) = u_0 (x - c_1 t) + \int_{0}^{t} \alpha\big(v(x + c_1 (s - t), s) - u(x + c_1 (s - t), s)\big) ds \,. \qquad (52)$$

Similarly we can do for the equation (47), hence we have

$$v(x,t) = v_0 (x - c_2 t) + \int_{0}^{t} \alpha\big(u(x + c_2 (s - t), s) - v(x + c_2 (s - t), s)\big) ds \qquad (53)$$

Another way we can solve two channels dissipation model : firstly by transformations and then by characteristic method. Transform $u(x,t)$ and $v(x,t)$ by

$$u(x,t) = e^{-\alpha t} U(x,t)$$
$$v(x,t) = e^{-\alpha t} V(x,t) \qquad (54)$$

so we have new initial value problem:

$$U_t (x,t) + c_1 U_x (x,t) = \alpha V(x,t)$$
$$V_t (x,t) + c_2 V_x (x,t) = \alpha U(x,t)$$
$$U(x,0) = u_0 (x)$$
$$V(x,0) = v_0 (x) \qquad (55)$$

The equations (53) are solved by characteristic method, we obtain:

$$U(x,t) = u_0 (x - c_1 t) + \int_{0}^{t} \alpha V(x + c_1 (s - t), s) ds \qquad (56)$$

$$V(x,t) = v_0 (x - c_1 t) + \int_{0}^{t} \alpha U(x + c_2 (s - t), s) ds \qquad (57)$$

or

$$u(x,t) = u_0 (x - c_1 t) e^{-\alpha t} + \int_{0}^{t} \alpha v(x + c_1 (s - t), s) e^{\alpha (s - t)} ds \qquad (58)$$

$$v(x,t) = v_0 (x - c_2 t) e^{-\alpha t} + \int_{0}^{t} \alpha u(x + c_2 (s - t), s) e^{\alpha (s - t)} ds \,. \qquad (59)$$

The equation (52)-(53) and (58)-(59) are called integral equation form of the initial value of the two channel dissipation model. It is very useful to prove the existence solution of initial value problem of two channel dissipation model.

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