

Automorphic Distributions, L-functions, and Voronoi Summation for $GL(3)$

Stephen D. Miller* and Wilfried Schmid†

June 8, 2003

1 Introduction

In 1903 Voronoi [40] postulated the existence of explicit formulas for sums of the form

$$(1.1) \quad \sum_{n \geq 1} a_n f(n),$$

for any “arithmetically interesting” sequence of coefficients $(a_n)_{n \geq 1}$ and every f in a large class of test functions, including characteristic functions of bounded intervals. He actually established such a formula when $a_n = d(n)$ is the number of positive divisors of n [41]. He also asserted a formula for

$$(1.2) \quad a_n = \#\{(a, b) \in \mathbb{Z}^2 \mid Q(a, b) = n\},$$

where Q denotes a positive definite integral quadratic form [42]; Sierpiński [38] and Hardy [14] later proved the formula rigorously. As Voronoi pointed out, this formula implies the bound

$$(1.3) \quad \left| \#\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 \leq x\} - \pi x \right| = O(x^{1/3})$$

for the error term in Gauss’ classical circle problem, improving greatly on Gauss’ own bound $O(x^{1/2})$. Though Voronoi originally deduced his formulas

*Supported by NSF grant DMS-0122799 and an NSF post-doctoral fellowship

†Supported in part by NSF grant DMS-0070714

from Poisson summation in \mathbb{R}^2 , applied to appropriately chosen test functions, one nowadays views his formulas as identities involving the Fourier coefficients of modular forms on $GL(2)$, i.e., modular forms on the complex upper half plane. A discussion of the Voronoi summation formula and its history can be found in our expository paper [26].

The main result of this paper is a generalization of the Voronoi summation formula to $GL(3, \mathbb{Z})$ -automorphic representations of $GL(3, \mathbb{R})$. Our technique is quite general; we plan to extend the formula to the case of $GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A})$ in the future. The arguments make heavy use of representation theory. To illustrate the main idea, we begin by deriving the well-known generalization of the Voronoi summation formula to coefficients of modular forms on $GL(2)$, stated below in (1.12–16). This formula is actually due to Wilton – see [16] – and is not among the formulas predicted by Voronoi. However, because it is quite similar in style one commonly refers to it as a Voronoi summation formula. We shall follow this tradition and regard our $GL(3)$ formula as an instance of Voronoi summation as well. The $GL(2)$ formula is typically derived from modular forms via Dirichlet series and Mellin inversion; see, for example, [8, 21]. We shall describe this connection with Dirichlet series later on in this introduction. Since we want to exhibit the analytic aspects of the argument, we concentrate on the case of modular forms invariant under $\Gamma = SL(2, \mathbb{Z})$. The changes necessary to treat the case of a congruence subgroup can easily be adapted from [8, 21], for example.

We consider a cuspidal, $SL(2, \mathbb{Z})$ -automorphic form Φ on the upper half plane $H = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. This covers two separate possibilities: Φ can either be a holomorphic cusp form, or – necessarily even – weight k ,

$$(1.4) \quad \Phi(z) = \sum_{n=1}^{\infty} a_n n^{(k-1)/2} e(nz) \quad (e(z) =_{\text{def}} e^{2\pi iz}),$$

or a cuspidal Maass form – i.e., $\Phi \in C^\infty(H)$, $y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi = -\lambda \Phi$ with $\lambda = \frac{1}{4} - \nu^2$, $\nu \in i\mathbb{R}$, and

$$(1.5) \quad \Phi(x + iy) = \sum_{n \neq 0} a_n \sqrt{|y|} K_\nu(2\pi|n|y) e(nx)$$

[23]. In either situation, Φ is completely determined by the distribution

$$(1.6) \quad \tau(x) = \sum_{n \neq 0} a_n |n|^{-\nu} e(nx),$$

with the understanding that in the holomorphic case we set both $a_n = 0$ for $n < 0$ and $\nu = -\frac{k-1}{2}$. One can also describe τ as a limit in the distribution topology: $\tau(x) = \lim_{y \rightarrow 0^+} \Phi(x + iy)$ when Φ is a holomorphic cusp form; the analogous formula for Maass forms is slightly more complicated [34]. As a consequence of these limit formulas, τ inherits automorphy from Φ ,

$$(1.7) \quad \tau(x) = |cx + d|^{2\nu-1} \tau\left(\frac{ax+b}{cx+d}\right), \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

This is the reason for calling τ the automorphic distribution attached to Φ . The regularity properties of automorphic distributions for $SL(2, \mathbb{R})$ have been investigated in [2, 22, 34], but these properties are not important for the argument we are about to sketch.

If $c \neq 0$ in (1.7), we can substitute $x - d/c$ for x , which results in the equivalent equation

$$(1.8) \quad \tau\left(x - \frac{d}{c}\right) = |cx|^{2\nu-1} \tau\left(\frac{a}{c} - \frac{1}{c^2x}\right).$$

We now integrate both sides of (1.8) against a test function g in the Schwartz space $\mathcal{S}(\mathbb{R})$. On one side we get

$$(1.9) \quad \begin{aligned} \int_{\mathbb{R}} \tau\left(x - \frac{d}{c}\right) g(x) dx &= \int_{\mathbb{R}} \sum_{n \neq 0} a_n |n|^{-\nu} e(nx - \frac{nd}{c}) g(x) dx \\ &= \sum_{n \neq 0} a_n |n|^{-\nu} e(-\frac{nd}{c}) \widehat{g}(-n). \end{aligned}$$

On the other side, arguing formally at first, we find

$$(1.10) \quad \begin{aligned} \int_{\mathbb{R}} |cx|^{2\nu-1} \tau\left(\frac{a}{c} - \frac{1}{c^2x}\right) g(x) dx &= \\ &= |c|^{2\nu-1} \int_{\mathbb{R}} |x|^{2\nu-1} \sum_{n \neq 0} a_n |n|^{-\nu} e\left(\frac{na}{c} - \frac{n}{c^2x}\right) g(x) dx \\ &= |c|^{2\nu-1} \sum_{n \neq 0} a_n |n|^{-\nu} e\left(\frac{na}{c}\right) \int_{\mathbb{R}} |x|^{2\nu-1} e\left(-\frac{n}{c^2x}\right) g(x) dx. \end{aligned}$$

To justify this computation, we must show that (1.8) can be interpreted as an identity of tempered distributions defined on all of \mathbb{R} . A tempered distribution, we recall, is a continuous linear functional on the Schwartz space

$\mathcal{S}(\mathbb{R})$, or equivalently, a derivative of some order of a continuous function having at most polynomial growth. Like any periodic distribution, τ is certainly tempered. In fact, since the Fourier series (1.6) has no constant term, τ can even be expressed as the n -th derivative of a *bounded* continuous function, for every sufficiently large $n \in \mathbb{N}$. This fact, coupled with a simple computation, exhibits $|cx|^{2\nu-1} \tau\left(\frac{a}{c} - \frac{1}{c^2x}\right)$ as an n -th derivative of a function which is continuous, even at $x = 0$. Consequently this distribution extends naturally from \mathbb{R}^* to \mathbb{R} . Using the cuspidality of Φ , one can show further that the identity (1.8) holds in the strong sense – i.e., the extension of $|cx|^{2\nu-1} \tau\left(\frac{a}{c} - \frac{1}{c^2x}\right)$ which was just described coincides with $\tau\left(x - \frac{d}{c}\right)$ even across the point $x = 0$. The fact that τ is the n -th derivative of a bounded continuous function, for all large n , can also be used to justify interchanging the order of summation and integration in the second step of (1.10). In any event, the equality (1.10) is legitimate, and the resulting sum converges absolutely. For details see the analogous argument in Section 5 for the case of $GL(3)$, as well as [27], which discusses the relevant facts from the theory of distributions in some detail.

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function which vanishes to infinite order at the origin, or more generally, a function such that $|x|^\nu f(x) \in \mathcal{S}(\mathbb{R})$. Then $g(x) = \int_{\mathbb{R}} f(t)|t|^\nu e(-xt) dt$ is also a Schwartz function, and $f(x) = |x|^{-\nu} \widehat{g}(-x)$. With this choice of g , (1.8–10) imply

$$\begin{aligned}
& \sum_{n \neq 0} a_n e(-nd/c) f(n) = \\
&= \sum_{n \neq 0} a_n e(na/c) \frac{|c|^{2\nu-1}}{|n|^\nu} \int_{x=-\infty}^{\infty} |x|^{2\nu-1} e\left(-\frac{n}{c^2x}\right) \int_{t=-\infty}^{\infty} f(t)|t|^\nu e(-xt) dt dx \\
&= \sum_{n \neq 0} a_n e(na/c) \frac{|c|^{2\nu-1}}{|n|^\nu} \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} |x|^{-2\nu-1} |t|^\nu f(t) e\left(-\frac{t}{x} - \frac{nx}{c^2}\right) dt dx \\
&= \sum_{n \neq 0} a_n e(na/c) \frac{|c|^{2\nu-1}}{|n|^\nu} \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} |x|^{-\nu} |t|^\nu f(xt) e\left(-t - \frac{nx}{c^2}\right) dt dx \\
&= \sum_{n \neq 0} a_n e(na/c) \frac{|c|}{|n|} \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} |x|^{-\nu} |t|^\nu f\left(\frac{xtc^2}{n}\right) e(-t - x) dt dx .
\end{aligned}$$

In this derivation, the integrals with respect to the variable t converge absolutely, since they represent the Fourier transform of a Schwartz function. The integrals with respect to x , on the other hand, converge only when $\operatorname{Re} \nu > 0$, but have meaning for all $\nu \in \mathbb{C}$ by holomorphic continuation.

So far, we have assumed only that a, b, c, d are the entries of a matrix in

$SL(2, \mathbb{Z})$, and $c \neq 0$. We now fix a pair of relatively prime integers a, c , with $c \neq 0$, and choose a multiplicative inverse \bar{a} of a modulo c :

$$(1.11) \quad a, c, \bar{a} \in \mathbb{Z}, \quad (a, c) = 1, \quad c \neq 0, \quad \bar{a}a \equiv 1 \pmod{c}.$$

Then there exists $b \in \mathbb{Z}$ such that $a\bar{a} - bc = 1$. Letting \bar{a}, b, c, a play the roles of a, b, c, d in the preceding derivation, we obtain the *Voronoi Summation Formula* for $GL(2)$:

$$(1.12) \quad \sum_{n \neq 0} a_n e(-na/c) f(n) = |c| \sum_{n \neq 0} \frac{a_n}{|n|} e(n\bar{a}/c) F(n/c^2).$$

Here a_n and ν have the same meaning as in (1.4–6), $f(x) \in |x|^{-\nu} \mathcal{S}(\mathbb{R})$, and

$$(1.13) \quad F(t) = \int_{\mathbb{R}^2} f\left(\frac{x_1 x_2}{t}\right) |x_1|^\nu |x_2|^{-\nu} e(-x_1 - x_2) dx_2 dx_1.$$

One can show further that this function F vanishes rapidly at infinity, along with all of its derivatives, and has identifiable potential singularities at the origin:

$$(1.14) \quad F(x) \in \begin{cases} |x|^{1-\nu} \mathcal{S}(\mathbb{R}) + |x|^{1+\nu} \mathcal{S}(\mathbb{R}) & \text{if } \nu \notin \mathbb{Z} \\ |x|^{1-\nu} \log|x| \mathcal{S}(\mathbb{R}) + |x|^{1+\nu} \mathcal{S}(\mathbb{R}) & \text{if } \nu \in \mathbb{Z}_{\leq 0} \end{cases}$$

[27, (6.58)]; the case $\nu \in \mathbb{Z}_{>0}$ never comes up. The formula (1.13) for F is meant symbolically, of course: it should be interpreted as a repeated integral, via holomorphic continuation, as in the derivation. Alternatively and equivalently, F can be described by Mellin inversion, in terms of the Mellin transform of f , as follows. Without loss of generality, we may suppose that f is either even or odd, say $f(-x) = (-1)^\eta f(x)$ with $\eta \in \{0, 1\}$. In this situation,

$$(1.15) \quad F(x) = \frac{\operatorname{sgn}(-x)^\eta}{4\pi^2 j} \int_{\operatorname{Re} s = \sigma} \pi^{-2s} \frac{\Gamma(\frac{1+s+\eta+\nu}{2}) \Gamma(\frac{1+s+\eta-\nu}{2})}{\Gamma(\frac{-s+\eta+\nu}{2}) \Gamma(\frac{-s+\eta-\nu}{2})} M_\eta f(-s) |x|^{-s} ds,$$

where $\sigma > |\operatorname{Re} \nu| - 1$ is arbitrary, and

$$(1.16) \quad M_\eta f(s) = \int_{\mathbb{R}} f(x) \operatorname{sgn}(x)^\eta |x|^{s-1} dx$$

denotes the signed Mellin transform. For details see section 5, where the $GL(3)$ analogues of (1.14–15) are proved.

If one sets $c = 1$ and formally substitutes the characteristic function $\chi_{[\epsilon, x+\epsilon]}$ for f in (1.12), one obtains an expression for the sum $\sum_{0 < n \leq x} a_n$; formulas of this type were considered especially useful in Voronoi's time. There is an extensive literature on the range of allowable test functions f . However, beginning in the 1930s, it became clear that "harsh" cutoff functions like $\chi_{[\epsilon, x+\epsilon]}$ are no more useful from a technical point of view than the type of test functions we allow in (1.12).

The Voronoi summation formula for $GL(2)$ has become a fundamental analytic tool for a number of deep results in analytic number theory, most notably to the sub-convexity problem for automorphic L -functions; see [18] for a survey, as well as [10, 21, 32]. In these applications, the presence of the additive twists in (1.12) – i.e., the factors $e(-na/c)$ on the left hand side – has been absolutely crucial. These additive twists lead to estimates for sums of modular form coefficients over arithmetic progressions. They also make it possible to handle sums of coefficients weighted by Kloosterman sums, such as $\sum_{n \neq 0} a_n f(n) S(n, k; c)$, which appear in the Petersson and Kuznetsov trace formulas [13, 32]. In view of the definition of the Kloosterman sum $S(m, k; c)$, which we recall in the statement of our main theorem below,

$$\begin{aligned}
 (1.17) \quad \sum_{n \neq 0} a_n f(n) S(n, k; c) &= \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} e(kd/c) \sum_{n \neq 0} a_n f(n) e(n\bar{d}/c) \\
 &= |c| \sum_{n \neq 0} \frac{a_n}{|n|} F(n/c^2) \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} e((k-n)d/c).
 \end{aligned}$$

The last sum over d in this equation is a *Ramanujan sum*, which can be explicitly evaluated; see, for example, [17, p. 55]. The resulting expression for $\sum_{n \neq 0} a_n f(n) S(n, k; c)$ can often be manipulated further.

We should point out another feature of the Voronoi formula that plays an important role in applications. Scaling the argument x of the test function f by a factor T^{-1} , $T > 0$, has the effect of scaling the argument t of F by the reciprocal factor T . Thus, if f approximates the characteristic function of an interval, more terms enter the left hand side of (1.12) in a significant way as the scaling parameter T tends to infinity. At the same time, fewer terms contribute significantly to the right hand side. This mechanism of lengthening the sum on one side while simultaneously shortening the sum on the other side is known as "dualizing". It helps detect cancellation in sums like $\sum_{n \leq x} a_n f(n) e(-na/c)$ and has become a fundamental technique in the subject.

We mentioned earlier that our main result is an analogue of the $GL(2)$ Voronoi summation formula for cusp forms on $GL(3)$:

1.18 Theorem. *Suppose that $a_{n,m}$ are the Fourier coefficients of a cuspidal $GL(3, \mathbb{Z})$ -automorphic representation of $GL(3, \mathbb{R})$, as in (5.9), with representation parameters λ, δ , as in (2.10). Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function which vanishes to infinite order at the origin, or more generally, a function on $\mathbb{R} - \{0\}$ such that $(\text{sgn } x)^{\delta_3} |x|^{-\lambda_3} f(x) \in \mathcal{S}(\mathbb{R})$. Then for $(a, c) = 1$, $c \neq 0$, $\bar{a}a \equiv 1 \pmod{c}$ and $q > 0$,*

$$\sum_{n \neq 0} a_{q,n} e(-na/c) f(n) = \sum_{d|cq} \left| \frac{c}{d} \right|^{1-\lambda_1-\lambda_2-\lambda_3} \sum_{n \neq 0} \frac{a_{n,d}}{|n|} S(q\bar{a}, n; qc/d) F\left(\frac{nd^2}{c^3q}\right),$$

where $S(n, m; c) =_{\text{def}} \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{nx+m\bar{x}}{c}\right)$ denotes the Kloosterman sum and, in symbolic notation,

$$F(t) = \int_{\mathbb{R}^3} f\left(\frac{x_1 x_2 x_3}{t}\right) \prod_{j=1}^3 ((\text{sgn } x_j)^{\delta_j} |x_j|^{-\lambda_j} e(-x_j)) dx_3 dx_2 dx_1.$$

This integral expression for F converges when performed as repeated integral in the indicated order – i.e., with x_3 first, then x_2 , then x_1 – and provided $\text{Re } \lambda_1 > \text{Re } \lambda_2 > \text{Re } \lambda_3$; it has meaning for arbitrary values of $\lambda_1, \lambda_2, \lambda_3$ by analytic continuation. If $f(-x) = (-1)^\eta f(x)$, with $\eta \in \{0, 1\}$, one can alternatively describe F by the identity

$$F(x) = \frac{\text{sgn}(-x)^\eta}{4\pi^{5/2}i} \int_{\text{Re } s = \sigma} \pi^{-3s} \left(\prod_{j=1}^3 i^{\delta'_j} \pi^{\lambda_j} \frac{\Gamma\left(\frac{s+1-\lambda_j+\delta'_j}{2}\right)}{\Gamma\left(\frac{-s+\lambda_j+\delta'_j}{2}\right)} \right) M_\eta f(-s) |x|^{-s} ds;$$

here $M_\eta f(s)$ denotes the signed Mellin transform (1.16), the $\delta'_j \in \{0, 1\}$ are characterized by the congruences $\delta'_j \equiv \delta_j + \eta \pmod{2}$, and σ is subject to the condition $\sigma > \max_j (\text{Re } \lambda_j - 1)$ but otherwise arbitrary. The function F is smooth except at the origin and decays rapidly at infinity, along with all its derivatives. At the origin, F has singularities of a very particular type, which are described in (5.30–33) below.

Only very special types of cusp forms on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$ have been constructed explicitly; these all come from the Gelbart-Jacquet symmetric square functorial lift of cusp forms on $SL(2, \mathbb{Z}) \backslash H$ [11], though non-lifted

forms are known to exist and are far more abundant [25]. When specialized to these symmetric square lifts, our main theorem provides a non-linear summation formula involving the coefficients of modular forms for $GL(2)$. The relation between the Fourier coefficients of $GL(2)$ -modular forms and the coefficients of their symmetric square lifts is worked out in [26, §5].

Our main theorem, specifically the resulting formula for the symmetric squares of $GL(2)$ -modular forms, has already been applied to a problem originating from partial differential equations and the Berry/Hejhal random wave model in Quantum Chaos. Let X be a compact Riemann surface and $\{\phi_j\}$ an orthonormal basis of eigenfunctions for the Laplace operator on X . A result of Sogge [39] bounds the L^p -norms of the ϕ_j in terms of the corresponding eigenvalues λ_{ϕ_j} , and these bounds are known to be sharp. However, in the case of $X = SL(2, \mathbb{Z}) \backslash H$ – which is non-compact, of course, and not even covered by Sogge’s estimate – analogies and experimental data suggest much stronger bounds [15, 31]: when the orthonormal basis $\{\phi_j\}$ consists of Hecke eigenforms, one expects

$$(1.19) \quad \|\phi_j\|_p = O(\lambda_{\phi_j}^\epsilon) \quad (\epsilon > 0, \quad 0 < p < \infty).$$

Sarnak and Watson [33] have announced (1.19) for $p = 4$, at present under the assumption of the Ramanujan conjecture for Maass forms, whereas [39] gives the bound $O(\lambda_{\phi_j}^{1/16})$ in the compact case, for $p = 4$. Their argument uses our Voronoi summation formula, among other ingredients. To put this bound into context, we should mention that a slight variant of (1.19) would imply the Lindelöf Conjecture: $|\zeta(1/2 + it)| = O(1 + |t|^\epsilon)$, for any $\epsilon > 0$ [31].

There is a close connection between L -functions and summation formulas. In the prototypical case of the Riemann ζ -function, the Poisson summation formula – which should be regarded as the simplest instance of Voronoi summation – not only implies, but is equivalent to analytic properties of the ζ -function, in particular its analytic continuation and functional equation. The ideas involved carry over quite directly to the $GL(2)$ Voronoi summation formula (1.12), but encounter difficulties for $GL(3)$.

To clarify the nature of these difficulties, let us briefly revisit the case of $GL(2)$. For simplicity, we suppose Φ is a holomorphic cusp form, as in (1.4). A formal computation shows that the choice of $f(x) = |x|^{-s}$ corresponds to $F(t) = R(s)|t|^s$ in (1.13), with

$$(1.20) \quad R(s) = i^k (2\pi)^{2s-1} \frac{\Gamma(1 - s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})}.$$

Inserting these choices of f and F into (1.12) results in the equation

$$(1.21) \quad \sum_{n=1}^{\infty} a_n e(-na/c) n^{-s} = R(s) |c|^{1-2s} \sum_{n=1}^{\infty} a_n e(n\bar{a}/c) n^{s-1},$$

which has only symbolic meaning because the regions of convergence of the two series do not intersect. We should remark, however, that the methods of our companion paper [27] can be used to make this formal argument rigorous. When $c = 1$, (1.21) reduces to the functional equation of the standard L -function $L(s, \Phi) = \sum_{n=1}^{\infty} a_n n^{-s}$. Taking linear combinations over the various $a \in (\mathbb{Z}/c\mathbb{Z})^*$ for a fixed $c > 1$ gives the functional equation for the multiplicatively twisted L -function

$$(1.22) \quad L(s, \Phi \otimes \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$$

with twist χ , which can be any primitive Dirichlet character mod c .

The traditional derivation of (1.12), in [8, 21] for example, argues in reverse. It starts with the functional equations for $L(s, \Phi)$ and expresses the left-hand side of the Voronoi summation formula through Mellin inversion,

$$(1.23) \quad \sum_{n=1}^{\infty} a_n f(n) = \frac{1}{2\pi i} \int_{\text{Re } s = \sigma} L(s, \Phi) Mf(s) ds, \quad Mf(s) = \int_0^{\infty} f(t) t^{s-1} dt,$$

with $\sigma > 0$. The functional equation for $L(s, \Phi)$ is then used to conclude $\sum_{n=1}^{\infty} a_n f(n) = \sum_{n=1}^{\infty} a_n F(n)$, where $MF(s) = r(1-s)Mf(1-s)$. To deal with additive twists, one applies the same argument to the multiplicatively twisted L -functions $L(s, \Phi \otimes \chi)$. A combinatorial argument makes it possible to express the additive character $e(-na/c)$ in terms of the multiplicative Dirichlet characters modulo c ; this not particularly difficult. An analogous step appears already in the classical work of Dirichlet and Hurwitz on the Dirichlet L -functions $\sum_{n=1}^{\infty} \chi(n) n^{-s}$. For $GL(3)$, the same reasoning carries over quite easily, but only until this point: the combinatorics of converting multiplicative information to additive information on the right hand side of the Voronoi formula becomes far more complicated. For one thing, the functional equation for the $L(s, \Phi \otimes \chi)$ only involves the coefficients $a_{1,n}$ and $a_{n,1}$, whereas the right hand side of the Voronoi formula involves also the other coefficients. It is possible to express all the $a_{n,m}$ in terms of the $a_{1,n}$ and $a_{n,1}$, but this requires Hecke identities and is a non-linear process. The Voronoi formula, on the other hand, is a purely additive, seemingly non-arithmetic statement about the $a_{n,m}$. In the past, the problem of converting

multiplicative to additive information was the main obstacle to proving a Voronoi summation formula for $GL(3)$. Our methods bypass this difficulty entirely by dealing with the automorphic representation directly, without any input from the Hecke action.

The Voronoi summation formula for $GL(3, \mathbb{Z})$ encodes information about the additively twisted L -functions $\sum_{n \neq 0} e(na/c) a_{n,q} |n|^{-s}$. It is natural to ask if this information is equivalent to the functional equations for the multiplicatively twisted L -functions $L(s, \Phi \otimes \chi)$. The answer to this question is yes: in section 6 we derive the functional equations for the $L(s, \Phi \otimes \chi)$, and in section 7, we reverse the process by showing that it is possible after all to recover the additive information from these multiplicatively twisted functional equations. It turns out that our analysis of the boundary distribution – concretely, the $GL(3)$ analogues of (1.7–10) – presents the additive twists in a form which facilitates conversion to multiplicative twists. Section 7 concludes with a proof of the $GL(3)$ converse theorem of [20]. Though this theorem has been long known, of course, our arguments provide the first proof for $GL(3)$ that can be couched in classical language, i.e., without adèles. To explain why this might be of interest, we recall that Jacquet-Langlands gave an adelic proof of the converse theorem for $GL(2)$ under the hypothesis of functional equations for all the multiplicatively twisted L -functions [19]. However, other arguments demonstrate that only a finite number of functional equations are needed [29, 44]. In particular, for the full-level subgroup $\Gamma = SL(2, \mathbb{Z})$, Hecke proved a converse theorem requiring the functional equation merely for the standard L -function. Until now it was not clear what the situation for $GL(3)$ would be. Our arguments demonstrate that automorphy under $\Gamma = GL(3, \mathbb{Z})$ is equivalent to the functional equations for all the twisted L -functions. Since the various twisted L -functions are generally believed to be analytically independent – their zeroes are uncorrelated [30], for example – our analysis comes close to ruling out a purely analytic proof using fewer than all the twists.

Our paper proves the Voronoi summation formula only for cuspidal forms, automorphic with respect to the full-level subgroup $\Gamma = GL(3, \mathbb{Z})$. It is certainly possible to adapt our arguments to the case of general level N , but the notation would become prohibitively complicated. For this reason, we intend to present an adelic version of our arguments in the future, which will also treat the case of $GL(n)$, and not just $GL(3)$. Extending our formula to non-cuspidal automorphic forms would involve some additional technicalities. We are avoiding these because summation formulas for Eisenstein series can

be derived from formulas for the smaller group from which the Eisenstein series in question is induced. In fact, the Voronoi summation formula for a particular Eisenstein series on $GL(3)$, relating to sums of the triple divisor function $d_3(n) = \#\{x, y, z \in \mathbb{N} \mid n = xyz\}$, has appeared in [1] and in [7], in a somewhat different form.

Some comments on the organization of this paper: in the next section we present the representation-theoretic results on which our approach is based, in particular the notion of automorphic distribution. Automorphic distributions for $GL(3, \mathbb{Z})$ restrict to $N_{\mathbb{Z}}$ -invariant distributions on the upper triangular unipotent subgroup $N \subset GL(3, \mathbb{R})$, and they are completely determined by their restrictions to N . We analyze these restrictions in section 3, in terms of their Fourier expansions on $N_{\mathbb{Z}} \backslash N$. Proposition 3.18 gives a very explicit description of the Fourier decomposition of distributions on $N_{\mathbb{Z}} \backslash N$; we prove the proposition in section 4. Section 5 contains the proof of our main theorem, i.e., of the Voronoi summation formula for $GL(3)$. The proof relies heavily on a particular analytic technique – the notion of a distribution vanishing to infinite order at a point, and the ramifications of this notion. Since the technique applies in other contexts as well, we are developing it in a separate companion paper [27]. We had mentioned already that we derive the functional equations for the L -functions $L(s, \Phi \otimes \chi)$ in section 6, using the results of the earlier sections, and that section 7 contains our proof of the Converse Theorem of [20].

It is a pleasure to thank James Cogdell, Dick Gross, Roger Howe, David Kazhdan, Peter Sarnak, and Thomas Watson for their encouragement and helpful comments.
