THE ORBIT METHOD FOR THE JACOBI GROUP

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1 Introduction

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. We may identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ (cf. [Vo3], Proposition 2.7). More precisely the real valued symmetric bilinear form $< \, , \, >: \mathfrak{g} \times \mathfrak{g} \longrightarrow$ defined by

$$< X, Y >= \mathrm{Re}\,\mathrm{tr}\,(XY), \quad X, Y \in \mathfrak{g} \tag{1.1}$$

is nondegenerate and hence there exists a G-equivariant linear isomorphism

$$\mathfrak{g}^* \longrightarrow \mathfrak{g}, \quad \lambda \mapsto X_\lambda$$

characterized by $\lambda(Y) =< X_\lambda, Y >$, $Y \in \mathfrak{g}$. Therefore the coadjoint G-orbits in $\mathfrak{g}^*$ may be identified with adjoint G-orbits in $\mathfrak{g}$. The philosophy of the orbit method says that we may attach the irreducible unitary representations of G to the coadjoint orbits in $\mathfrak{g}^*$. Historically the orbit method that was first initiated by A.A. Kirillov (cf. [K]) early in the 1960s in a real nilpotent Lie group worked beautifully. Thereafter the orbit method was extended nicely to a solvable Lie group of type I by Auslander and Kostant (cf. [A-K]). Their proof was based on the existence of complex polarizations satisfying a positivity condition. Unfortunately Kirillov's work fails to be generalized in some ways to the case of compact Lie groups or semisimple Lie groups. Relatively simple groups like $SL(2, \mathbb{R})$ have irreducible unitary representations that do not correspond to any symplectic homogeneous space. Conversely, P. Torasso [T] found that the double cover of $SL(3, \mathbb{R})$ has a homogeneous symplectic manifold corresponding to no unitary representations. The orbit method for reductive Lie groups is a kind of a philosophy but not a theorem. Many large families of orbits correspond in comprehensible ways to unitary representations, and provide a clear geometric picture of these representations. The coadjoint orbits for a reductive Lie group are classified into three kinds of orbits, namely, hyperbolic, elliptic and nilpotent ones. The hyperbolic orbits are related to the unitary representations obtained by the parabolic induction and on the other hand, the elliptic ones are related to the unitary representations obtained by the cohomological induction. However, we still have no idea of attaching unitary representations to nilpotent orbits. It is known that there are only finitely many nilpotent orbits. In a certain case, some nilpotent orbits are corresponded to the so-called *unipotent representations*. For instance, a minimal nilpotent orbit is attached to a minimal representation. In fact, the notion of unipotent representations is not still well defined. The investigation of unipotent representations is now under way.

In this paper, we study the orbit method for the Jacobi group G^J. The Jacobi group G^J is a semidirect product of the symplectic group and the Heisenberg group. For a detail, we refer to Section 3 in this paper. G^J is *not* a reductive Lie group. The real-valued symmetric bilinear form $< \, , \, >: \mathfrak{g}^J \times \mathfrak{g}^J \longrightarrow \mathbb{R}$ defined by $< X, Y >= \mathrm{tr}\,(XY),\ \ X, Y \in \mathfrak{g}^J$ (cf. (1.1)) is highly degenerate. Therefore in the Jacobi group G^J, we can not identify the Lie algebra $\mathfrak{g}^J$ of G^J with its dual $(\mathfrak{g}^J)^*$. We may not expect nice properties which are obtained in the reductive case. In fact, there are infinitely many nilpotent G^J-orbits and there is no correspondence like the Kostant-Sekiguchi correspondence occurring in the reductive case. The paper is organized as follows. In Section 2, we review the orbits and the Kostant-Sekiguchi correspondence for the group $SL(2,\mathbb{R})$. In Section 3, we investigate the adjoint nilpotent G^J-orbits in $\mathfrak{g}^J$ explicitly. In particular, we provide an injective mapping from the set of nilpotent G^J-orbits in $\mathfrak{g}$ to the set of nilpotent $K^J_{\mathbb{C}}$-orbits in $\mathfrak{p}^J_{\mathbb{C}}$. In the final section, we investigate the coadjoint G^J-orbits in $(\mathfrak{g}^J)^*$ explicitly. We review the unitary representations of $SL(2,\mathbb{R})$ and G^J. And then we attach to these orbits unitary representations of G^J .

NOTATIONS: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. We denote by $\mathbb{R}^\times$ and $\mathbb{C}^\times$ the set of nonzero real numbers and the set of nonzero complex numbers respectively. We denote by $\mathbb{Z}^+$ (resp. $\mathbb{Z}_{\geq 0}$) the set of all positive (resp. nonnegative) integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. We denote the identity matrix of degree k by I_k.