Compatible Decompositions and Block Realizations of Finite Metrics

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Abstract

Given a metric D defined on a finite set X, we define a finite collection $\mathcal{D}$ of metrics on X to be a **compatible decomposition** of D if any two distinct metrics in $\mathcal{D}$ are linearly independent (considered as vectors in $\mathbb{R}^{X \times X}$), $D = \sum_{d \in \mathcal{D}} d$ holds, and there exist points $x, x' \in X$ for any two distinct metrics d, d' in $\mathcal{D}$ such that $d(x, y)\, d'(x', y) = 0$ holds for every $y \in X$. In this paper, we show that such decompositions are in one-to-one correspondence with (isomorphism classes of) **block realizations** of D, that is, graph realizations G of D for which G is a **block graph** and for which every vertex in G not labelled by X has degree at least 3 and is a **cut point** of G. This generalizes a fundamental result in phylogenetic combinatorics that states that a metric D defined on X can be realized by a **tree** if and only if there exists a compatible decomposition $\mathcal{D}$ of D such that all metrics $d \in \mathcal{D}$ are split metrics, and lays the foundation for a more general theory of metric decompositions that will be explored in future papers.

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