Computing virtual cut points

1 Introduction

Let X denote a finite set with n elements, and $D : X \times X \to \mathbb{R}_{\geq 0} : (x, y) \mapsto xy$ a metric that is defined on X. Let

$$T(D) := \{f \in \mathbb{R}^X : f(x) = \sup(xy - f(x) : y \in X) \text{ holds for all } x \in X\}$$

denote the tight span of D. A map $f \in T(D)$ is a *virtual cut point* of D if there exists a bipartition of the support $\mathrm{supp}(f) := \{x \in X : f(x) > 0\}$ of f into two non-empty subsets A and B such that $ab = f(a) + f(b)$ holds for all $a \in A$ and $b \in B$.

Such maps have been studied in the context of *optimal realizations* of finite metrics in [5, 8, 9, 10]. In [7], it has been shown recently that a map $f \in T(D)$ is a virtual cut point of D if and only if $T(D) \setminus \{f\}$ is disconnected, and that there exists a canonical one-to-one correspondence between (i) the connected components of $T(D) \setminus \{f\}$ and (ii) those of the graph $\Gamma_f := (\mathrm{supp}(f), E_f)$ with vertex set $\mathrm{supp}(f)$ and edge set

$$E_f = \{\{x, y\} \in \binom{\mathrm{supp}(f)}{2} : f(x) + f(y) > xy\}.$$

In [6], an algorithm for computing the virtual cut points of D has been proposed. A straight forward implementation of this algorithm has run time $O(n^6)$. Here we present an alternative implementation having run time $O(n^3)$ (Theorem 4.1).