Wavelet-based extraction of coherent structures to analyze and compute Navier-Stokes equations

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IMS, Singapore, 2004
What is turbulence?

- Highly nonlinear regime characteristic of fluid motions governed by the Navier-Stokes equations:

\[
\partial_t \vec{\omega} + \vec{V} \cdot \nabla \vec{\omega} - \vec{\omega} \cdot \nabla \vec{V} - \nu \nabla^2 \vec{\omega} = \nabla \times \vec{F}
\]

with
\[
\vec{\omega} = \nabla \times \vec{V} \quad \text{and} \quad \nabla \cdot \vec{V} = 0
\]

plus initial conditions and boundary conditions

- Turbulent flows involve a large number of degrees of freedom excited on a wide range of scales, called the inertial range,

- The flow evolution is non conservative due to viscous dissipation in strong shear regions.

The Reynolds number \( Re \) is the ratio between the nonlinear advection term and the linear dissipation term:

\[ \text{turbulence} \Rightarrow Re >> 1 \Rightarrow \text{nonlinear behaviour} >> \text{linear behaviour} \]
2D turbulent flow: evolution of vorticity
Vorticity field in wavelet space

The red interface corresponds to the threshold $T$: coefficients $> T$ (below the interface) are coherent, coefficients $< T$ (above) are incoherent.
Evolution of the interface in wavelet space

The isosurface corresponds to the threshold $T$
'In the last decade we have experienced a conceptual shift in our view of turbulence. For flows with strong velocity shear, or other organizing characteristics, many now feel that the spectral description has inhibited fundamental progress. The next "El Dorado" lies in the mathematical understanding of coherent structures in weakly dissipative fluids: the formation, evolution and interaction of metastable vortex-like solutions of nonlinear partial differential equations…'

Norman Zabusky, Physics Today, 1984

Replace the Fourier representation by the wavelet representation to keep track of the dynamics of coherent structures

J. Fluid Mech., 206, 1989
Interpretation of the turbulent cascade

'The terms "scale of motion" or "eddy of size l " appear repeatedly in the treatments of the inertial range. One gets an impression of little, randomly distributed whirls in the fluid, with the cascade process consisting of the fission of the whirls into smaller ones, after the fashion of Richardson’s poem. This picture seems drastically in conflict with what can be inferred about the qualitative structures of high-Reynolds number turbulence from laboratory visualization techniques and from the application of the Kelvin's circulation theorem'.

Robert Kraichnan, J. Fluid Mech., 1962

We should find a new interpretation of the turbulent cascade, taking into account the nonlinear dynamics of Navier-Stokes equations and the formation of coherent vortices in regions of strong shear

analyze, filter and compute Navier-Stokes equations in wavelet space
Orthogonal wavelet transform

Wavelet analysis

\[ \tilde{f}_{ji} = \langle \psi_{ji} \left| f \right. \rangle \quad \text{with} \quad \psi_{ji}(x) = 2^{j/2} \psi(2^j x - i) \]

Wavelet synthesis

\[ f(x) = \sum_{j=0}^{+\infty} \sum_{i=-\infty}^{+\infty} \langle \psi_{ji} \left| f \right. \rangle \cdot \psi_{ji}(x). \]

A signal sampled on N points is analyzed and synthetized in CN operations, e.g. C=12 for Coifman wavelets.
Wavelets are bandpass filters with $\Delta k/k = \text{constant}$.
Iterative Wavelet Denoising (IWD)

Signal $f = \text{Coherent signal} + \text{Incoherent noise}$

Iterative algorithm:

1. The signal $f$, sampled on $N$ points, is projected on a wavelet basis to obtain the coefficients $\tilde{f}$.

2. Set $n=1$ and $\tilde{f}_1 = \tilde{f}$.

3. Compute the threshold $T_n = \left(2 \langle \tilde{f}_1 \rangle^2 \ln N \right)^{\frac{1}{2}}$.

4. The incoherent coefficients $\tilde{f}_I$ are those for which $|\tilde{f}| < T_n$.

5. If $T_n \neq T_{n-1}$ go to 3. and set $n=n+1$, else go to 6.

6. The coherent signal $f_C$ is reconstructed from $|\tilde{f}| \geq T_n$.
   The incoherent noise $f_I$ is reconstructed from $|\tilde{f}| < T_n$. 
2D academic example  (SNR = 10 dB)

PDF of denoised signal and noise

PDF of wavelet coefficients
Extraction of coherent vortices using IWD

0.2 % of coefficients
99.8 % of kinetic energy
93.6 % of enstrophy

Coherent vorticity
0.2 % N

99.8 % of coefficients
0.2 % of kinetic energy
6.4 % of enstrophy

Incoherent vorticity
99.8% N

Total vorticity
N=512²
Energy spectrum

Coherent $k^{-5}$ scaling, i.e. long-range correlation

Incoherent $k^{-1}$ scaling, i.e. enstrophy equipartition
PDF of vorticity

$P(\omega)$

Coherent skewed exponential

Incoherent unskewed Gaussian
CVS filtering of the flow evolution

DNS of vorticity

CVS filtering

coherent
- vorticity
- velocity

incoherent
- vorticity
- velocity

mixing and reaction in the total flow

mixing and reaction in the coherent part

mixing and reaction in the incoherent part
CVS decomposition using IWD

- 0.2% of coefficients
- 99.8% of kinetic energy
- 93.6% of enstrophy

Coherent vorticity
0.2% N

99.8% of coefficients
0.2% of kinetic energy
6.4% of enstrophy

Incoherent vorticity
99.8% N

Total vorticity
N=512²

QuickTime™ et un décompresseur codec YUV420 sont requis pour visionner cette image.
Passive scalar advection

Coherent flow

Total flow

Incoherent flow
Passive scalar advection

by the total flow

by the coherent flow

by the incoherent flow
Evolution of the concentration variance

\[ \sigma(0) - \sigma(t) \]

Coherent flow

\[ t^{3/2} \]

anomalous diffusion due to transport by vortices

Incoherent flow

\[ t^{1/2} \]

classical diffusion
Advection of tracer particles

by the total flow

by the coherent flow

by the incoherent flow

Trapping and transport by vortices

Diffusion by Brownian motion
Time correlation of the Lagrangian velocity

- Coherent flow: long time correlation
- Incoherent flow: short time correlation

Graph showing the correlation function $R^L(t)$ for total, coherent, and incoherent flows over time $t$. The graph indicates a decay in the correlation with increasing time for both coherent and incoherent flows.
Results for 2D turbulence

- With only 0.2% of the wavelet coefficients one retains
  - 99.8% of the kinetic energy,
  - 93.6% of enstrophy,
  - all coherent vortices of the flow.

- Mixing in the coherent part reproduces mixing in the total flow, for both passive scalars and tracer particles,
  due to trapping and transport by the coherent vortices.

- Mixing in the incoherent part shows a diffusive behaviour, similar to Brownian motion, which corresponds to turbulent dissipation.
The experimental data have been obtained using PIV (Particle Image Velocimetry) by:

Jori E. Ruppert-Felsot
Olivier Praud
Eran Sharon
Harry L. Swinney

Center for Nonlinear Dynamics,
University of Texas at Austin
• Rotate up to 1.0 Hz
• Mechanical pumping of fluid through hexagonal array of sources and sinks

• 100 μm seed particles

• Use PIV to calculate vorticity fields

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Experimental setup

![Diagram of experimental setup](image)
Vorticity observed by PIV

Coherent vorticity

Total vorticity

Incoherent vorticity
PDF of vorticity

$P(\omega)$ vs $\omega [s^{-1}]$
Extraction of coherent vortices with CVS filter

- 3% of coefficients
- 99% of kinetic energy
- 75% of enstrophy

- 97% of coefficients
- 0.6% of kinetic energy
- 25% of enstrophy

Coherent vorticity

Incoherent vorticity

Total vorticity
Extraction of coherent vortices with LES filter

- 3% of coefficients
- 99% of kinetic energy
- 64% of enstrophy

Large scale vorticity

- 97% of coefficients
- 0.9% of kinetic energy
- 36% of enstrophy

Small scale vorticity

- Total vorticity
Energy spectrum

Coherent
$k^{-5}$ scaling, i.e. long-range correlation

Incoherent
$k^{+1}$ scaling, i.e. energy equipartition
PDF of velocity

CVS filter
Gaussian incoherent flow,
i.e. easy to model

LES filter
non-Gaussian small scales,
i.e. difficult to model
Results for 3D turbulence

- With only 3% of the wavelet coefficients CVS retains
  - 99% of the kinetic energy,
  - 75% of enstrophy,
  - all coherent vortices of the flow field.

- We observe
  long-range correlation for the coherent flow,
  energy equipartition and Gaussian velocity
  for the incoherent flow, in contrast to LES.

- We conjecture that discarding the incoherent flow is sufficient to model turbulent dissipation.
Coherent Vortex Simulation (CVS)

- Projection of vorticity onto an orthogonal wavelet basis.
- Coherence extraction using the IWD algorithm.
- Reconstruction of the coherent vorticity.
- Computation of the coherent velocity using Biot-Savart’s law.
- Addition of a security zone in wavelet space.
- Integration of Navier-Stokes of in the reduced wavelet basis.

*Phys. Fluids, 11(8), 1999*
*Flow, turbulence and combustion, 66(4), 2001*
*Phys. Fluids 15 (10), 2003*
'Kolmogorov's theory has been counter-productive. It is OK for light or sound scattering by turbulent flows, but it is unuseful for the main lines of turbulence. In turbulence you have long range forces, and it is difficult to extrapolate from Boltzmann's gas, which has short range forces. Therefore I am uneasy about Reynolds equations. As long as we will not be able to predict the drag on a sphere or the pressure drop in a pipe from continuous, incompressible and Newtonian assumptions, without any other complications, i.e. from first principles, we will not have make it!'.

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Combine CVS and volume penalization to compute flows with boundaries