Cupping Computably Enumerable Degrees in the Difference Hierarchy

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A set $D$ is d.c.e. if there are c.e. sets $B$ and $C$ such that $D = B - C$. $D$ is the difference of two c.e. sets.

A Turing degree is d.c.e. if it contains a d.c.e. set.

Every c.e. degree is d.c.e.

(Arlsanov) Every nonzero d.c.e. degree is cuppable. (cf. In R, there exist noncuppable degrees.)
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D.c.e. sets and d.c.e. degrees

- A set $D$ is d.c.e. if there are c.e. sets $B$ and $C$ such that $D = B \setminus C$. $D$ is the difference of two c.e. sets.
- A Turing degree is d.c.e. if it contains a d.c.e. set.
- Every c.e. degree is d.c.e..
- (Arslanov) Every nonzero d.c.e. degree is cuppable.
  (cf. In $\mathcal{R}$, there exist noncuppable degrees.)
Proof

• Given $A$ (c.e. set) incomputable. We will construct an incomplete d.c.e. set $D$ and a computable functional $\Gamma$ such that $K = \Gamma^{A,D}$, where $K$ is a fixed creative set.
Proof

- Given $A$ (c.e. set) incomputable. We will construct an incomplete d.c.e. set $D$ and a computable functional $\Gamma$ such that $K = \Gamma^{A,D}$, where $K$ is a fixed creative set.

$\mathcal{R}: K = \Gamma^{A,D}$.

$\mathcal{P}_e: E \neq \Phi^D_e$. 
\( R: \)
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- If \( n \) enters \( K \), and we want to rectify \( \Gamma^{A,D}(n) \) (defined as 0) at stage \( s \), we put \( \gamma(n)[s] \) (or smaller) into \( D \), to undefine \( \Gamma^{A,D}(n) \).
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- If \( A \) changes blow \( \gamma(n)[s] \) after stage \( s \), then we can take \( \gamma(n)[s] \) out of \( D \) since this \( A \) change can undefine \( \Gamma^{A,D}(n) \).
$P$: 

Step 1: Choose $x$ and $k$. If $K$ changes below $k$, then we start from the beginning, except that we keep $k$ the same.

Such a refresh (or reset) procedure can happen at most $k$ many times.

$k$ is called a "threshold" of $P$.

Step 2: Wait for $D(x)$ to converge to 0.

Step 3: Put $\circ(k)$ into $D$. Go back to step 2, and simultaneously, wait for $A$ to change below $\circ(k)$.

Step 4: Take $\circ(k)$ out of $D$ and put $x$ into $E$. 
$\mathcal{P}$:

- **Step 1**: Choose $x$ and $k$.
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  ✷ Such a refresh (or reset) procedure can happen at most $k$ many times.
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- **Step 2**: Wait for \( \Phi^D(x) \) to converge to 0.
$P$: 

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  If $K$ changes below $k$, then we start from the beginning, except that we keep $k$ the same.
  
  ◊ Such a refresh (or reset) procedure can happen at most $k$ many times.
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- Reach step 4 eventually. Again, $\mathbf{P}$ is satisfied.
- Stop at step 3 infinitely many times. Then $\mathbf{A}$ is computable, which can be called a pseudo-outcome of $\mathbf{P}$. 


Nondensity Theorem
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There is a maximal incomplete d.c.e. degree.
From nondensity to almost universal cupping

- An incomplete d.c.e. degree is almost universal cupping if it cups each c.e. degree not below it to $0'$. 
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- Between almost universal cupping degree and $0'$, there are no c.e. degrees.
From nondensity to almost universal cupping

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- There is no d.c.e. universal cupping degree.
An incomplete d.c.e. degree is **almost universal cupping** if it cups each c.e. degree not below it to $0'$. 

Between almost universal cupping degree and $0'$, there are no c.e. degrees.

There is no d.c.e. universal cupping degree.

Maximal incomplete degrees are almost universal cupping.
Theorem 1

Almost universal cupping degrees exist.
Proof

We will construct a d.c.e. set $A$ such that for each c.e. set $W$, either $A$ cups $W$ to $K$ or $A$ computes $W$. 

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\[ R_e : K = \Gamma_e^{A,W_e} \text{ or } W_e = \Delta_e^A. \]
Proof

We will construct a d.c.e. set \( A \) such that for each c.e. set \( W \), either \( A \) cups \( W \) to \( K \) or \( A \) computes \( W \).

\[ \mathcal{R}_e: K = \Gamma_{e}^{A,W_e} \text{ or } W_e = \Delta_{e}^{A}. \]

\[ \mathcal{P}_e: E \neq \Phi_{e}^{A}. \]
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$\mathcal{R}_e: K = \Gamma_{e,We}^A$ or $W_e = \Delta_e^A$.

$\mathcal{P}_e: E \neq \Phi_e^A$.

• Compare with Arslanov’s requirements.
Interactions between strategies

Similar to the proof of Arslanov’s cupping theorem. (Oracle $A$ in $\mathcal{A}$ seems not necessary in this special case.)

Consider the interactions of two $P$ strategies.

Two ways to get around the obstacle:

1. Make $A$ !-c.e. and universal cupping (Li, Song and Wu)
2. Make $A$ d.c.e. but $\mathcal{A}$ is now necessary.
Interactions between strategies

• One $\mathcal{R}$ and one $\mathcal{P}$. 

Similar to the proof of Arslanov’s cupping theorem. (Oracle $\mathcal{A}$ in $\mathcal{C}$ $\mathcal{A}$ seems not necessary in this special case.)

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Two ways to get around the obstacle.

§ Make $\mathcal{A} \not\in$-c.e. and universal cupping (Li, Song and Wu)

§ Make $\mathcal{A}$ d.c.e. but $\mathcal{C} \mathcal{A}$ is now necessary.
Interactions between strategies

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  Similar to the proof of Arslanov’s cupping theorem. (Oracle $A$ in $\Delta^A$ seems not necessary in this special case.)

- Consider the interactions of two $\mathcal{P}$ strategies.
- Two ways to get around the obstacle.
  - Make $A$ $\omega$-c.e. and universal cupping (Li, Song and Wu)
  - Make $A$ d.c.e. but $\Delta^A$ is now necessary.
A nonzero c.e. degree has plus-cupping property if any nonzero c.e. degree below it can be cupped to 0.

If $c$ cups $b$ to 0 then $b$ is called a cupping partner of $c$.

How many cupping partners are needed in this definition?

Answer: infinite.
A nonzero c.e. degree has plus-cupping property if any nonzero c.e. degree below it can be cupped to $0'$. How many cupping partners are needed in this definition?

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Three alternative approaches
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• (Slaman)

There are c.e. degrees $a > 0$, $b$, $c$, with $b \nleq c$ such that $b$ cups any nonzero c.e. degree below $a$ above $c$. 
Three alternative approaches

• (Slaman)
There are c.e. degrees \( a > 0, b, c \), with \( b \nsubseteq c \) such that \( b \) cups any nonzero c.e. degree below \( a \) above \( c \).

• (Slaman)
There are c.e. degrees \( a, b > 0 \) such that for any \( c \leq a \), if \( c \nsubseteq b \), then \( c \cup b = 0' \).
Theorem 2 (Plus-cupping for d.c.e.)

There are a c.e. degree $a > 0$ and an incomplete d.c.e. degree $d$ such that $d$ cups each nonzero c.e. degree below $a$ to 0.
Theorem 2 (Plus-cupping for d.c.e.)

There are a c.e. degree $a > 0$ and an incomplete d.c.e. degree $d$ such that $d$ cups each nonzero c.e. degree below $a$ to $0'$. 
Proof
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\[ Q_e: A \neq \Phi_e; \]
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$M_e: E \neq \Phi_e^D;$
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\( M_e: E \neq \Phi_e^D; \)

\( N_e: W_e = \Phi_e^A \implies K = \Gamma_e^{W_e,D} \) or \( W_e \) is computable.
Theorem 1+2

There are d.c.e. degrees $a; d$ such that if $c$ is a nonzero c.e. degree below $a$ then $c[d] = 0$, and if $c$ is a c.e. degree not below $a$ then $c[a] = 0$.

Consider the interactions of these two arguments.
There are d.c.e. degrees $a, d$ such that if $c$ is a nonzero c.e. degree below $a$ then $c \cup d = 0'$, and if $c$ is a c.e. degree not below $a$ then $c \cup a = 0'$. 
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- Consider the interactions of these two $0'''$ arguments.
Compare with Li-Yi’s cupping
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(Li and Yi) There are two d.c.e. degrees b, d such that any nonzero c.e. degree cups one of them to $0'$. 
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(Li and Yi) There are two d.c.e. degrees b, d such that any nonzero c.e. degree cups one of them to 0'.

\( \mathcal{R} \): \( W \) is computable, or \( W \) cups \( B \) to \( K \), or \( W \) cups \( D \) to \( K \).
Compare with Li-Yi’s cupping

(Li and Yi) There are two d.c.e. degrees $b$, $d$ such that any nonzero c.e. degree cups one of them to $0'$. 

$R$: $W$ is computable, or $W$ cups $B$ to $K$, or $W$ cups $D$ to $K$. 

Theorem 1+2 implies Li and Yi’s cupping. Extra properties.
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Li and Yi’s cupping implies Theorem 2.
More consequences
More consequences

- Arslanov’s cupping theorem
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- Arslanov’s cupping theorem
- Downey’s diamond embedding
More consequences

- Arslanov’s cupping theorem
- Downey’s diamond embedding
- $N_5$ embedding
More cupping
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(1) There are intervals of d.c.e. degrees containing exactly one c.e. degree.
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(3) These c.e. degrees can be low.
Questions

- In Theorem 1+2, can we have the almost universal cupping there maximal?
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• In Theorem 1+2, can we have the almost universal cupping there maximal?

• How to define computably enumerable degrees in the $\Delta^0_2$ degrees?
Thank you!