Some properties of c.e. reals in the \textit{sw-degrees}

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Downey, Hirschfeldt, and Laforet introduced a measure of relative complexity call *sw-reducibility* (strong weak truth table reducibility).
Introduction and definitions

Downey, Hirschfeldt, and Laforet introduced a measure of relative complexity call *sw-reducibility* (strong weak truth table reducibility).

**Definition**

A set $A$ is nearly computably enumerable if there is a computable approximation $\{A_s\}_{s \in \mathbb{W}}$ such that $A(x) = \lim_s A_s(x)$ for all $x$ and $A_s(x) > A_{s+1}(x) \Rightarrow \exists y < x (A_s(y) < A_{s+1}(y))$. 
A real $\alpha$ is computably enumerable (c.e) if $\alpha = 0.\chi_A$ where $A$ is a nearly c.e. set. A real $\alpha$ is strongly computably enumerable (strongly c.e.) if $\alpha = 0.\chi_A$ where $A$ is a c.e. set.

Definition

Let $A, B \subseteq \mathbb{N}$. We say that $B$ is strongly weak truth table reducible (sw-reducible) to $A$, and write $B \leq_{\text{sw}} A$, if there is a Turning reduction $\Gamma$ such that $B = \Gamma A$ and the use $\gamma(x) \leq x + c$ for some constant $c$. For reals $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, we say that $\beta$ is sw-reducible to $\alpha$, and write $\beta \leq_{\text{sw}} \alpha$ if $B \leq_{\text{sw}} A$.
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The \textit{sw} degrees have a number of nice aspects

For instance, Downey, Hirschfeldt, and Nies proved \textit{sw}-reducibility satisfies Solovay property and

\textbf{Theorem (Downey, Hirschfeldt, Laforte)}

\textit{Let }$\alpha$\textit{ and }$\beta$\textit{ be c.e. reals such that }$
\lim \inf_n H(\alpha \upharpoonright n) - H(\beta \upharpoonright n) = \infty$. \textit{Then }$\beta \leq_{sw} \alpha$.

Furthermore if $\alpha$ is a c.e. real which is noncomputable, then there is a noncomputable strongly c.e. real $\beta \leq_{sw} \alpha$, and this is not true in general, for $\leq_S$. 
coincidence of s-reducibility and sw-reducibility on strong c.e. reals
Theorem

If $\beta$ is strongly c.e. and $\alpha$ is c.e. then $\alpha \leq_{sw} \beta$ implies $\alpha \leq_S \beta$. 

coincidence of s-reducibility and sw-reducibility on strong c.e. reals

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If \( \alpha \) is strongly c.e. and \( \beta \) is c.e. then \( \alpha \leq_{S} \beta \) implies \( \alpha \leq_{sw} \beta \).
However, we still are interested in sw-reducibility since it has some nice properties and it is helpful for studying Turing-degrees by exploring the sw-degrees. Further, we may study the structure of c.e. reals in the sw-degrees.
Let $A$ be a nearly c.e. set. The sw-canonical c.e. set $A^*$ associated with $A$ is defined as follows. Begin with $A^*_0 = \emptyset$. For all $x$ and $s$, if $x \not\in A_s$ and $x \in A_{s+1}$, or $x \in A_s$ and $x \not\in A_{s+1}$, then for the least $j$ with $<x,j> \not\in A^*_s$, put $<x,j>$ into $A^*_{s+1}$.

Theorem (Downey, Hirschfeldt, Laforte)
If $A$ is nearly c.e. and noncomputable then there is a noncomputable c.e. set $A^* \leq_{sw} A$. Hence there are no minimal sw-degrees of c.e. reals.

Theorem (Downey, Hirschfeldt, Laforte)
There exist nearly c.e. sets $A$ and $B$ such that for all nearly c.e. $W \geq_{sw} A$, $B$ there is a nearly c.e. $Q$ with $A$, $B \leq_{sw} Q$ but $W \not\leq_{sw} Q$. Thus the sw-degrees of c.e. reals do not form an uppersemilattice.
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If $A$ is nearly c.e. and noncomputable then there is a noncomputable c.e. set $A^* \leq_{sw} A$. Hence there are no minimal sw-degrees of c.e. reals.

Theorem (Downey, Hirschfeldt, Laforte)
There exist nearly c.e. sets $A$ and $B$ such that for all nearly c.e. $W \geq_{sw} A, B$ there is a nearly c.e. $Q$ with $A, B \leq_{sw} Q$ but $W \not\leq_{sw} Q$. Thus the sw-degrees of c.e. reals do not form an uppersemilattice.
Yu Liang and Ding Decheng pointed out that we cannot characterize randomness by $sw$-reducibility by proving that there is no a largest c.e. $sw$-degree.

**Theorem (Yu and Ding)**

*There is no $sw$-complete c.e. real. Even more, there is a pair of c.e. reals for which there is no c.e. real above both of them respect to $sw$-reducibility.*
Theorem (Fan and Lu)

Let \( \{\alpha_e\}_{e \in \omega} \) be an effective enumeration of strongly c.e. reals. Then there are strongly c.e. reals \( \beta_0, \beta_1 \) such that \( \beta_0 \not\leq_{sw} \alpha_e \) or \( \beta_1 \not\leq_{sw} \alpha_e \) for every \( \alpha_e \).

Proof

\[ R_{e,i} : \Phi_i^{\alpha_e} \neq \beta_0 \lor \Psi_i^{\alpha_e} \neq \beta_1, \]

where \( \phi_i(x) \leq x + i \) and \( \psi_i(x) \leq x + i \).

Pick a large number \( k_{e,i} \) for \( R_{e,i} \) such that \( k_{e,i} > e, i \) and \( k_{e,i} > 3k_{e',i'} \) for all \( e' < e \) or \( e = e', i' < i \).

We only put numbers between \( k_{e,i} \) and \( 3k_{e,i} \) into \( B \) or \( C \) for \( R_{e,i} \).
our results of c.e. reals in the sw-degrees

**Theorem (Fan and Lu)**

Let \( \{\alpha_e\}_{e \in \omega} \) be an effective enumeration of strongly c.e. reals. Then there is a c.e. real \( \beta \) such that \( \alpha_e \leq_{sw} \beta \) for every \( \alpha_e \).

**Proof**

\[ R_e : \Gamma_e^\beta = \alpha_e, \]

where \( \Gamma_e \) is defined by us such that \( \gamma_e(x) \leq x + e + 3 \).

1) Check whether there exist some \( R_e \) such that \( \alpha_e(x) \) changes.
2) Choose the least \( e \leq s \) such that
\[ \exists (x \leq s)[\alpha_{e,s+1}(x) \neq \alpha_{e,s}(x)] \}.
3) Set \( \beta_{s+1} = \beta_s \upharpoonright (x + e + 3) + 2^{-(x+e+3)} \).
Definition (Yu)

A c.e. real $\alpha$ is sw-cuppable if there is a c.e. real $\beta$ such that there is no c.e. real above both of them respect to sw-reducibility.
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Theorem (Yu)
There exists a sw-cuppable c.e. real.

Theorem (Fan and Lu)
For any c.e. real $\alpha$, there exists a c.e. real $\beta$ such that $\beta$ is sw-cuppable and $\alpha \leq_{sw} \beta$. 
Theorem (Fan and Lu)

Let \( \{\alpha_e\}_{e \in \omega} \) be an effective enumeration of c.e. reals. Then there is a strongly c.e. real \( \beta_0 \) and a c.e. real \( \beta_1 \) such that \( \beta_0 \not\leq_{sw} \alpha_e \) or \( \beta_1 \not\leq_{sw} \alpha_e \) for every \( e \).

Proof

\[ R_e : \phi_e^{\alpha_e} \neq \beta_0 \lor \psi_e^{\alpha_e} \neq \beta_1, \]

For simplicity, we assume that \( \phi^\alpha(x) = x \) and \( \phi^\alpha(x) = x \).
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Let \( \{\alpha_e\}_{e \in \omega} \) be an effective enumeration of c.e. reals. Then there is a strongly c.e. real \( \beta_0 \) and a c.e. real \( \beta_1 \) such that \( \beta_0 \not\leq_{sw} \alpha_e \) or \( \beta_1 \not\leq_{sw} \alpha_e \) for every \( e \).

Proof

\[
R_e : \Phi_e^{\alpha_e} \neq \beta_0 \lor \Psi_e^{\alpha_e} \neq \beta_1,
\]

Lemma
Given \((n, k)\), there is a strongly c.e. real \( \beta_0 \), a c.e. real \( \beta_1 \) and \( l \) such that there exists a function \( \Gamma : [0, 1) \times [0, 1) \rightarrow R \) satisfies \( \Gamma(\beta_0 \upharpoonright l, \beta_1 \upharpoonright l) \geq n \) and \( \beta_0 \upharpoonright k = 0 \). Moreover, \( \beta_0, \beta_1 \) and \( l \) can be computed uniformly from \((n, k)\).

For simplicity, we assume that \( \phi^{\alpha}(x) = x \) and \( \phi^{\alpha}(x) = x \).
The proof of the lemma is divided into two cases: (1) the induction on $n$; (2) the induction on $k$.

Now consider the case for the induction on $n$. Fixed $n$, assume that $\Gamma(\beta_0, i, k \upharpoonright l_i, k, \beta_1, i, k \upharpoonright l_i, k) \geq i$ and $\beta_0, i, k \upharpoonright k = 0$ for every $i \leq n$, $k \in \mathbb{N}$. Let $l_{n+1,0}$ be equal to $(l_n, l_{n,0} + 1)$.

**Step 1.** Imitate our programme for putting numbers into $A_{n, l_n,0} \upharpoonright l_n, l_{n,0}, B_{n, l_n,0} \upharpoonright l_n, l_{n,0}$, and do the similar action on the natural number between 2 and $l_{n+1,0}$.

Note that $\Gamma(\beta_0, n, l_n,0 \upharpoonright l_n, l_{n,0}, \beta_1, n, l_n,0 \upharpoonright l_n, l_{n,0}) \geq n$, $\beta_0, n, l_n,0 \upharpoonright l_n,0 = 0$. It must be $\Gamma(\beta_0, n, l_n,0 \upharpoonright l_n, l_{n,0}, \beta_1, n, l_n,0 \upharpoonright l_n, l_{n,0}) = n$ and $\beta_0, n, l_n,0 \upharpoonright l_n,0 = 0$ at some stage $t$. Hence, at stage $t$, $\Gamma_t(\beta_0, n+1,0, t \upharpoonright l_n, l_{n,0}, \beta_1, n+1,0, t \upharpoonright l_n, l_{n,0}) = n/2$, and $A_{n+1,0, t} \upharpoonright l_n,0 + 1 = 0$, $B_{n+1,0, t} \upharpoonright 1 = 0$. 
Step 2. At stage $t + 1$, let $A_{n+1,0}$ active and $B_{n+1,0}$ waiting, set $A_{n+1,0,t+1}(1) = 1$, which forces 
$\Gamma_{t+1}(\beta_{0,n+1,0,t+1} \uparrow l_n,l_n,0, \beta_{1,n+1,0,t+1} \uparrow l_n,l_n,0)$ equal to $n/2 + 1/2$. At stage $t + 2$, let $A_{n+1,0}$ be waiting and $B_{n+1,0}$ active, set $B_{n+1,0,t+2}(1) = 1, B_{n+1,0,t+2}(q) = 0$ ($q > 1$), which forces $\Gamma_{t+2}(\beta_{0,n+1,0,t+2} \uparrow l_{n+1,0}, \beta_{1,n+1,0,t+2} \uparrow l_{n+1,0}) = n/2 + 1$.

Step 3. Imitate the programme of the changes of $A_{n,0} \uparrow l_{n,0}, B_{n,0} \uparrow l_{n,0}$. Note that $A_{n+1,0,t+2}(x) = B_{n+1,0,t+2}(x) = 0$ ($2 \leq x \leq l_{n,0} + 1$), do the following similar actions. Imitate the programme.

Note that $\Gamma(\beta_{0,n,0} \uparrow l_{n,0}, \beta_{1,n,0} \uparrow l_{n,0}) \geq n$. The effect of the changes on $[2, l_{n,0}]$ of $A_{n+1,0}$ and $B_{n+1,0}$ induces $\Gamma(\beta_{0,n+1,0} \uparrow l_{n+1,0}, \beta_{1,n+1,0} \uparrow l_{n+1,0}) \geq n + 1$. 

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Next consider the case for the induction on $k$. Fix $(n, k)$, assume that $\Gamma(\beta_0,i,j \uparrow l_{i,j}, \beta_1,i,j \uparrow l_{i,j}) \geq i$ and $\beta_0,i,j \uparrow j = 0$ for every $i \leq n$ or $j \leq k$. We can win by controlling $A_{n,k+1} \uparrow l_{n,k+1}$, $B_{n,k+1} \uparrow l_{n,k+1}$ as follows. Let $l_{n,k+1}$ be equal to $l_{n-1},l_{n,k} + 1$.

**Step 1.** Imitate the programme of the changes of $A_{n-1,l_{n,k}} \uparrow l_{n-1,l_{n,k}}, B_{n-1,l_{n,k}} \uparrow l_{n-1,l_{n,k}}$.

Note that $\Gamma(\beta_0,n-1,l_{n,k} \uparrow l_{n-1,l_{n,k}}, \beta_1,n-1,l_{n,k} \uparrow l_{n-1,l_{n,k}}) \geq n - 1$, $\beta_{0,n-1,l_{n,k}} \uparrow l_{n,k} = 0$. It must be

$\Gamma(\beta_{0,n-1,l_{n,k}} \uparrow l_{n-1,l_{n,k}}, \beta_{1,n-1,l_{n,k}} \uparrow l_{n-1,l_{n,k}}) = n + 1$, $\beta_{0,n-1,l_{n,k}} \uparrow l_{n,k} = 0$ at some stage $t$. Hence, at stage $t$, $\Gamma_t(\beta_{0,n,k+1,t}, \beta_{1,n,k+1,t}) = (n - 1)/2$, and $A_{n,k+1,t} \uparrow l_{n,k,t} + 1 = 0$. 
Step 2. At stage $t + 1$, let $A_{n,k+1}$ waiting and $B_{n,k+1}$ active, set
$B_{n,k+1,t+2}(1) = 1$, $B_{n,k+1,t+2}(q) = 0 \ (q > 1)$, which forces
$\Gamma_{t+1}(\beta_{0,n,k+1,t+1} \uparrow l_{n,k+1}, \beta_{1,n,k+1,t+1} \uparrow l_{n,k+1}) = n/2$.

Step 3. Imitate the programme of the changes of
$A_{n,k} \uparrow l_{n,k}, B_{n,k} \uparrow l_{n,k}$. Here $A_{n,k+1,t+1}(x) = B_{n,k+1,t+1}(x) = 0$
$(2 \leq x \leq l_{n,k+1})$.

Note that $\Gamma(\beta_{0,n,k} \uparrow l_{n,k}, \beta_{1,n,k} \uparrow l_{n,k}) \geq (n - 1)$, $\beta_{0,n,k} \uparrow k = 0$.

The effect of the changes on $[2, l_{n,k} + 1]$ of $A_{n,k+1}$ and $B_{n,k+1}$
induces $\Gamma(\beta_{0,n,k+1} \uparrow l_{n,k+1}, \beta_{1,n,k+1} \uparrow l_{n,k+1}) \geq n$ and
$\beta_{0,n,k+1} \uparrow (k + 1) = 0$.

Since the construction is effective, $\beta_{0,n,k}$ is strongly c.e. and
$\beta_{1,n,k+1}$ is c.e. for every $n, k$. 
The main Theorem in the progress

Theorem
There exists a maximal c.e. reals in the sw-Degrees.

Corollary
There are $\aleph_0$ incomparable maximal c.e. reals in the sw-Degrees.

Theorem
For any noncomputable c.e. real $\alpha$, there exist a c.e. real $\beta$ such that $\beta \not\leq_{sw} \alpha$ and $\alpha \leq_T \beta$. 
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The proof of the theorem

It suffices to build a c.e. real $\alpha$ to meet the following requirements:

$$R_{<e,n>} : \alpha = \Phi^{\beta_e}_e \Rightarrow \exists \Gamma (\Gamma^\alpha = \beta_e)$$

where each $\{\Phi_e, \beta_e\}_{e \in \omega}$ is an enumeration of sw-procedures and c.e. reals with use $\phi_e(x) \leq x + n(n \in \omega)$.

Without loss of generality, suppose that $\beta_e$ is less than 0.1.
The special programm of the theorem

For any $\Phi_e$, our aim is to make $\alpha \neq \Phi^{\beta_e}$ or to define a function $\Gamma$ such that $\Gamma^\alpha = \beta_e$.

Assume that when we put some number $(\leq l(e, n))$ into $\alpha$ at expansionary stage, $\beta[(\phi_e(x) + 1)$ changes at the greatest position, i.e. the change is the slowest.

We assume that in digital expansion of $\beta$, there are infinite 1.
The strategy for $n = 0$

1. Wait for an expansionary stage when the first 1 appears in digital expansion of $\beta$, say at the position $m < l(e, n)$. Then we let $\alpha(m - 1)$ change to 1.
2. Wait for next expansionary stage and once we find it, $\beta(m_1)$ must change to 1. Then we let $\alpha(m - 2)$ change to 1 and wait for next expansionary stage.
Repeating the above strategy until $\beta$ have to be ready to change at position 1, by our assumption, this is impossible. Hence we win.
The strategy for $n = 1$

1. Wait for an expansionary stage when the first 1 appears in digital expansion of $\beta$, say at the position $m < l(e, n)$. Then we let $\alpha(m - 1)$ change to 1.
2. Wait for next expansionary stage and once we find it, $\beta(m - 1)$ must change. We let $\alpha(m - 2)$ change to 1 and wait for next expansionary stage. Repeating the above strategy until $\beta$ have to be ready to change at position 1, by our assumption, this is impossible. Hence we win.
The strategy for \( n = 2 \)

\[
\begin{array}{ccc}
\beta & 100 & 100 & 100 \\
\alpha & 100 & 100 & 100 \\
\beta & 100 & 101 & 0 \\
\alpha & 100 & 101 & 1 \\
\beta & 101 & 000 & 0 \\
\alpha & 101 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{ccc}
\beta & 100 & 100 & 110 \\
\alpha & 100 & 101 & 000 \\
\beta & 100 & 110 & 0 \\
\alpha & 100 & 111 & 0 \\
\beta & 100 & 000 & 0 \\
\alpha & 100 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{ccc}
\beta & 100 & 000 & 0 \\
\alpha & 100 & 000 & 0 \\
\end{array}
\]

Similarly we can get

\[
\begin{array}{ccc}
\beta & 11 & 000 & 0 \\
\alpha & 11 & 000 & 0 \\
\beta & 100 & 000 & 0 \\
\alpha & 100 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{ccc}
\beta & 100 & 000 & 0 \\
\alpha & 100 & 000 & 0 \\
\end{array}
\]
The strategy for $n = 2$

<table>
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<th>$\beta$</th>
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<th>$\beta$</th>
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<th>100</th>
<th>110</th>
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<td>000</td>
<td>0</td>
<td>$\Rightarrow$</td>
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</tbody>
</table>

Similarly we can get
Note that

\[
\begin{array}{c|c|c|c}
\beta & 10 & 000 & 0 \\
\alpha & 11 & 000 & 0 \\
\end{array}
\]

can move left forever by using 1 with lifting 1 if it only meet 1 with lifting 1. We call such case 11.

Using 11, the above case can change to

\[
\begin{array}{c|c|c|c}
\beta & 0110 & 000 & 0 \\
\alpha & 1000 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 100 & 000 & 0 \\
\alpha & 111 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 0110 & 000 & 0 \\
\alpha & 1000 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 100 & 000 & 0 \\
\alpha & 101 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 100 & 000 & 0 \\
\alpha & 110 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 1100 & 000 & 0 \\
\alpha & 1111 & 000 & 0 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\beta & 10000 & 000 & 0 \\
\alpha & 10000 & 000 & 0 \\
\end{array}
\]
The strategy for $n$

1. Wait for an expansionary stage, say $s_0$ when in $\beta$, the number of 1 is $\geq 2^{n-1} + 1$. Suppose that the position of the last 1 in $\beta$ is $t_0$.

2. Creating a situation such that we can apply $(n-1)$-strategy from next expansionary stage.
   a) then add 1 to the position $t_0 - 1$ to the $\alpha$ to force $\beta$ change at $t_0 - 1 + n$.
   b) Waiting for next expansionary stage when in $\beta$, there appear a new 1, say at position $t_1$, then add 1 to the position $t_1 - 1$ to the $\alpha$ to force $\beta$ change at $t_1 - 1 + n$.
   c) Repeating until we can get more than $2^{n-2} + 1$ new good 1 with lifting $n-1$ position after stage $s_0$.

Now the numbers on $\alpha$ and $\beta$ are:

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$010\ldots01$</th>
<th>$0\ldots0$</th>
<th>$010\ldots01$</th>
<th>$\ldots$</th>
<th>$010\ldots01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$100\ldots00$</td>
<td>$0\ldots0$</td>
<td>$100\ldots00$</td>
<td>$\ldots$</td>
<td>$100\ldots00$</td>
<td></td>
</tr>
</tbody>
</table>
to find the first fixed pair

| $\beta$ | 0100·01 | 0·00 | $\Rightarrow$ | $\beta$ | 011·001 | 0·00 |
| $\alpha$ | 1000·00 | 0·0 | $\Rightarrow$ | $\alpha$ | 110·000 | 0·0 |

| $\beta$ | 011·111 | 0·00 | $\Rightarrow$ | $\beta$ | 100·000 | 0·00 |
| $\alpha$ | 111·100 | 0·00 | $\Rightarrow$ | $\alpha$ | 111·110 | 0·00 |

This is called the first fixed pair. Note that it corresponds to

| $\beta$ | 0100·01 | 0·00 |
| $\alpha$ | 1000·00 | 0·00 |
to find the ability of the first fixed pair

when this first fixed point meet a block, we will prove that it can takeover the block, i.e.,

\[
\begin{array}{c|c|c|c}
\beta & 0 & 100 \cdots 00 & 0 \cdots 00 \\
\alpha & 1 & 111 \cdots 11 & 0 \cdots 00 \\
\end{array}
\]

injure it, it changes to

\[
\begin{array}{c|c|c|c}
\beta & 00 & 10 \cdots 01 & 0 \cdots 00 \\
\alpha & 10 & 00 \cdots 00 & 0 \cdots 00 \\
\end{array} \Rightarrow \begin{array}{c|c|c|c}
\beta & 01 & 10 \cdots 00 & 0 \cdots 00 \\
\alpha & 11 & 11 \cdots 10 & 0 \cdots 00 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\beta & 00 & 110 \cdots 01 & 0 \cdots 00 \\
\alpha & 10 & 000 \cdots 00 & 0 \cdots 00 \\
\end{array} \Rightarrow \begin{array}{c|c|c|c}
\beta & 01 & 1100 \cdots 00 & 0 \cdots 00 \\
\alpha & 11 & 1111 \cdots 10 & 0 \cdots 00 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\beta & 1 & 000 \cdots 00 & 0 \cdots 00 \\
\alpha & 1 & 111 \cdots 11 & 0 \cdots 00 \\
\end{array}
\]
look for the last fixed pair

the last fixed pair corresponds to

\[
\begin{array}{c|c|c|c}
\beta & 0 & 1 \cdot \cdot \cdot 1 & 0 \cdot \cdot \cdot 00 \\
\alpha & 1 & 0 \cdot \cdot \cdot 0 & 0 \cdot \cdot \cdot 00 \\
\end{array}
\]

By induction suppose that this is

\[
\begin{array}{c|c|c|c}
\beta & 100 \cdot \cdot \cdot 00 & 0 \cdot \cdot \cdot 00 \\
\alpha & a & 0 \cdot \cdot \cdot 00 \\
\end{array}
\]

when this last fixed pair meet a lifting \( n \)-number, i.e.,

\[
\begin{array}{c|c|c|c|c}
\beta & 1 & 100 \cdot \cdot \cdot 00 & 0 \cdot \cdot \cdot 00 \\
\alpha & 1 & a & 0 \cdot \cdot \cdot 00 \\
\end{array}
\]

applying this last fixed pair,

\[
\begin{array}{c|c|c|c|c}
\beta & 1 & 11 \cdot \cdot \cdot 11 & p & 0 \cdot \cdot \cdot 00 \\
\alpha & 1 & 11 \cdot \cdot \cdot 11 & q & 0 \cdot \cdot \cdot 00 \\
\end{array}
\]

injure it

\[
\begin{array}{c|c|c|c|c}
\beta & 1 & 00 \cdot \cdot \cdot 00 & 0 \cdot \cdot \cdot 00 \\
\alpha & 1 & 000 \cdot \cdot \cdot 0 & 0 \cdot \cdot \cdot 00 \\
\end{array}
\]

Note that there are \( 2^{n-1} \) fixed pairs.
6) Applying the above result repeatedly. Then we can force $\beta$ bigger enough. If we can applying it infinitely, then we can prove that $\beta \geq 0.1$, which is a contradiction. That is,

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0.0 1</th>
<th>11⋯11</th>
<th>0⋯00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0. 01</td>
<td>11⋯11</td>
<td>0⋯00</td>
</tr>
</tbody>
</table>

then we can get

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0.1 0</th>
<th>00⋯00</th>
<th>0⋯00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.10</td>
<td>00⋯00</td>
<td>0⋯00</td>
</tr>
</tbody>
</table>
1. Wait for the first expansionary stage, say $s_1$. Then compute $\Psi(e, H, s_1)$.

(1) If $\Psi(e, H, s_1) = 1$, then we do nothing and go to next $\sigma$.

(2) If $\Psi(e, H, s_1) = 0$, then from this stage we wait for a stage such that either $\Psi(e, H) = 1$ or there are $2^{n-1} + 1 + 2^{n-2} + 1 + \cdots + 2^{2-1} + 1$ times new 1 appear in $\beta$. 
2. If later at the next expansionary stage after $s_1$ there are 
$2^{n-1} + 1 + 2^{n-2} + 1 + \cdots + 2^{2-1} + 1$ times new 1 appear in $\beta$, 
then define (or redefine) $\Gamma^\alpha(x) = \beta_e(x)$ for any $x \leq l(e, n)$ with 
use $\gamma(x) = x + C$.

From this stage, at every expansionary stage, we should define and 
redefine $\Gamma^\alpha = \beta_e$. That is, if we find that $\beta(x)$ change to be 1 at 
some position $x$ some expansionary stage and $\Gamma^\alpha$ do not know, 
then we put some number $\leq \gamma(x)$ into $\alpha$, and initialise all 
strategies with lower priority.

Since we have got the prepared data, we can make a disagreement.
Suppose that $\sigma_0$ and $\sigma_1$ work on $R_0$-strategy and $R_1$-strategy respectively. And $\sigma_0 \subseteq \sigma_1$.

The $R_1$-strategy is:

1. Wait for the first expansionary stage, say $s_1$. Then compute $\Psi(1, H, s_1)$.
   (1) If $\Psi(1, H, s_1) = 1$, then we do nothing and go to next $\sigma$.
   (2) If $\Psi(1, H, s_1) = 0$, then from this stage we wait for a stage such that either $\Psi(e, H) = 1$ or there are $2^{n_1+r-1} + 1 + 2^{n_1+r-2} + 1 + \cdots + 2^{2-1} + 1$ times new $1$ appear in $\beta_1$. 

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Some properties of c.e. reals in the sw-degrees
2. If at the next expansionary stage, say $s_2$ there are $2^{n+r-1} + 1 + 2^{n+r-2} + 1 + \cdots + 2^{2-1} + 1$ times new 1 appear in $\beta$, then define (or redefine) $\Gamma^\alpha(x) = \beta_e(x)$. From $s_2$, at every expansionary stage, we should define and redefine $\Gamma^\alpha = \beta_1$.

Let $x_0 = \min\{\gamma_0(y) | \gamma_0(y) \text{ wants to enter into } \alpha\}$.
$x_1 = \min\{\gamma_1(y) | \gamma_1(y) \text{ wants to enter into } \alpha\}$.

If $x_0 < x_1$, then use the programme given above to make disagreement and initialise $R_1$.

If $x_0 \geq x_1$, then put $x_0$ into $\alpha$, apply the programme given above from the position $x_0$. If it make a disagreement, then we win. If not, then $x_1$ will eventually be put into $\alpha$.

Note that we delay putting $x_1$ into $\alpha$ here.