The orig. construction of the Q-curv. seeks to imitate

Yamabe eqn. in dim. $n \geq 3$ $\longrightarrow$

Gauss curv. prescription eqn. (GCP) in dim. 2.

It’s closely related to the GJMS operators $P_m$. To some extent, the construction can go either way ($P$'s to $Q$'s or $Q$'s to $P$'s). There are now constructions of $Q$ that are genuinely different than the original one we’ll discuss here (in which $Q$ comes from the GJMS series).

Let everything as acting on functions (0-densities) on an $n$-dimensional manifold $M$. 
The {\bf Graham-Jenne-Mason-Sparling} (GJMS) operators [{\it J. London Math. Soc. 1992}] were built using the {\bf Fefferman-Graham ambient construction}, and by careful analysis of the construction, have the properties in the following (redundant) list. Here \( n \) is not nec. even.

- \( P_m \) exists for \( m \) even and \( m - n \notin 2\mathbb{Z}^+ \).
- \( P_m = \Delta^{m/2} + \text{LOT} \).
- \( P_m \) is formally self-adjoint.
- For \( f \in C^\infty(M) \), under a conformal change of metric
  \[
  \hat{g} = e^{2\omega} g, \quad \omega \in C^\infty(M),
  \]
  we have the conformal covariance relation
  \[
  \hat{P}_m f = e^{-\frac{n+m}{2}\omega} P_m (e^{\frac{n-m}{2}\omega} f).
  \]
Alternatively, $P_m$ gives rise to a conformally invariant operator $P_m : \mathcal{E}[-(n - m)/2] \to \mathcal{E}[-(n + m)/2]$.

$P_m$ has a polynomial expression in $\nabla$ and the Riemann tensor (actually the Ricci tensor, according to a recent result of Graham) in which all coefficients are rational in the dimension $n$.

Gover and Peterson, CMP 2003 show that there’s an expression in which the only poles are given by factors $(n - 2)(n - 4) \cdots (n - m + 2)$ in the denominators of these rational functions.

On flat $\mathbb{R}^n$, $P_m = \Delta^{n/2}$. 


• $P_m$ has the form

$$\delta S_m d + \frac{n - m}{2} Q_m,$$

where $Q_m$ is a local scalar invariant, and $S_m$ is an operator on 1-forms of the form

$$(d\delta)^{m/2-1} + \text{LOT} \text{ or } \Delta^{m/2-1} + \text{LOT}.$$  

All the formulas mentioned above are universal.

Note that $P_m$ is unable to detect changes in the $(d\delta)^{m/2-1}$ term in the principal part of $S_m$.

**Remark** $P_m$ gives rise to a $Q_m$ in an elementary way (just take $P_m1$) when $m \neq n$. But the really important $Q$ is $Q = Q_n$. 
Remark \( P_2 \) is the conformal Laplacian
\[
Y = \frac{\delta d}{\Delta} + \frac{n-2}{4(n-1)} K.
\]
This makes
\[
Q_2 = \frac{K}{2(n-1)} =: J,
\]
(the Schouten scalar).

Here’s an intuitive approach (more formal approach later) to constructing the Q-curvature. The Yamabe eqn. is
\[
\left( \Delta + \frac{n-2}{2} J \right) u = \frac{n-2}{2} \hat{J} u \frac{(n+2)/(n-2)}{2},
\]
where
\[
\hat{g} = e^{2\omega} g, \quad \omega \in C^\infty(M), \quad u := e^{(n-2)\omega/2}.
\]
The GCP eqn. is
\[
\Delta \omega + J = \hat{J} e^{2\omega} \quad (n = 2).
\]
We get GCP from the Yamabe eqn. by slipping in a gratuitous 1,

\[ \Delta \left( e^{(n-2)\omega/2} - 1 \right) + \frac{n-2}{2} J e^{(n-2)\omega/2} \]

\[ = \frac{n-2}{2} \hat{J} e^{(n+2)\omega/2}, \]

dividing by \((n-2)/2\), and eval. at \(n = 2\).

Similarly, take the higher-order Yamabe equation based on the **GJMS operators**, 

\[ \left( \delta S_m d + \frac{n-m}{2} Q_m \right) u = \frac{n-m}{2} \hat{Q}_m u^{(n+m)/(n-m)}, \]

where

\[ u = e^{(n-m)\omega/2} \ (n \notin \{m, m-2, m-4, ..., 2, 0\}), \]

\[ S_m = (d\delta)^{m/2-1} + \text{LOT}. \]
We slip in the gratuitous 1,

\[
\delta S_{md}(e^{(n-m)\omega/2} - 1) + \frac{n-m}{2} Q_m e^{(n-m)\omega/2}
\]

\[
= \frac{n-m}{2} \hat{Q}_m e^{(n+m)\omega/2},
\]

divide by \((n-m)/2\), and evaluate at \(n = m\):

\[
P\omega + Q = \hat{Q} e^{n\omega}.
\]

That is, we define \(Q\) from the GJMS op. series, as

\[
\left[\frac{2P_m 1}{n-m}\right]_{n=m}.
\]

This construction of \(Q\) immediately gives its unusual linear conformal change law.

Switching to a density viewpoint (more later on this), we have a conformally invariant operator \(P: \mathcal{E}[0] \to \mathcal{E}[-n]\), and \(Q \in \mathcal{E}[-n]\) satisfying

\[
\hat{Q} = Q + P\omega.
\]
For example, GCP looks like

\[ \hat{J} = J + \Delta \omega \]

for \( J \) viewed as a \((-2)\)-density.

In hindsight, we have answered the

**Question**: Is there a higher (even) dimensional generalization of the exponential class Gauss curvature prescription problem \( \hat{J} = J + \Delta \omega \)?

But the Q-curvature also plays other important roles in conformal geometry, in that:
• Its integral has total metric variation the Fefferman-Graham obstruction tensor;

• It provides the geometric expression of the exponential class Beckner-Moser-Trudinger inequality;

• It provides the main term in Polyakov formulas for the quotient of functional determinants or torsion quantities, at 2 conformally related metrics;

• It provides one of the important terms in volume renormalization asymptotics at conformal infinity [Fefferman-Graham, MRL 2002].
**Background:** The Einstein (divergence free Ricci) tensor $E$ is the total metric variation of the scalar curvature. This means that if we take a smooth curve of metrics $g(\varepsilon)$, denote $(d/d\varepsilon)|_{\varepsilon=0}$ by a $\bullet$, and suppose

$$g(0) = g, \quad g^\bullet = h,$$

then

$$\left( \int K \, dv_{g} \right)^\bullet = \int h^{ab} E_{ab} dv_{g}.$$  

This is how the Einstein-Hilbert action leads to the Einstein equation.

**Background:** In dimension 4, the Bach tensor $B$ is the total metric variation of $|C|^2$, where $C$ is the Weyl tensor.
**Question:** In general even dimension $n$, the Fefferman-Graham tensor $O_{ab}$ is the obstruction to the power series construction of the ambient metric assoc. to a conformal structure. Is $O_{ab}$ the total metric variation of anything natural?

**Answer:** Yes, the Q-curvature, according to Graham-Hirachi, math.DG/0405068. In fact, for the $(-n)$-density version $Q$ of the Q-curv.,

$$
\left( \int Q \right)^* = \int h^{ab} O_{ab} dv_g.
$$

This is sensible at least when $h$ has compact support.
**Background:** Beckner’s [Ann. M. 1993] generalization, from $S^2$ to $S^n$, of the celebrated Moser-Trudinger inequality, says that with normalized measure on the sphere (and taking $n$ even for simplicity),

$$\log \int_{S^n} e^{n(\omega - \bar{\omega})} \leq \frac{n}{2(n-1)!} \int_{S^n} \omega P \omega,$$

where

$$P = \Delta \{\Delta + n - 2\} \{\Delta + 2(n - 3)\} \cdot \{\Delta + 3(n - 4)\} \cdots \{\Delta + \frac{n}{2} \left(\frac{n}{2} - 1\right)\}.$$ 

Equality holds iff there is a diffeomorphism $h$ of $S^n$ for which $h^* g_{\text{round}} = e^{2\omega} g_{\text{round}}$.

**Remark:** See [Branson, JFA 1987] for an early sighting of the operator $P$. 

Remark: (2D) Moser-Trudinger is
\[
\log \int_{S^2} e^{2(\omega - \bar{\omega})} \leq \int_{S^2} \omega \Delta \omega.
\]
But in higher dim., note that $P$ is more delicate than just $\Delta^{n/2}$. Closely related inequalities figure in de Branges’ resolution of the Bieberbach conjecture (the Lebedev-Mihlin inequality), and Perelman’s work on the Poincaré conjecture (Gross’ logarithmic Sobolev inequality).

These are sharp endpoint derivatives of borderline Sobolev imbeddings, or duals of such.

Question: Is there an expression of Beckner’s inequality that just involves some local invariant? Something like the soln. of the Yamabe problem, which realizes the Sobolev imbedding $L^2_1 \hookrightarrow L^{2n/(n-2)}$ as the problem of mimimizing $\int K$ over volume 1 metrics?
**Answer:** For vol. 1 metrics $\hat{g} = e^{2\omega}g$, where $g = g_{\text{round}},$

$$0 \leq \int_{S^n} \omega(\hat{Q} + Q).$$

**Remark:** The borderline Sob. imbeddings are $L^2_r \hookrightarrow L^{2n/(n-2r)}$, and the Beckner-MT edge of the borderline is $L^2_{n/2} \hookrightarrow e^L$.

**Remark:** This gives a glimpse of an interesting 2-metric functional on a conformal class,

$$Q(\hat{g}, g) = \frac{1}{2} \int \omega(\hat{Q} + Q).$$

This is alternating, and satisfies the cocycle condition

$$Q(\hat{g}, g) = Q(\hat{g}, \hat{g}) + Q(\hat{g}, g)$$

for iterated conformal changes. Here $\hat{g} = e^{2\eta}\hat{g}$, $\hat{g} = e^{2\omega}g$, where $\omega$ and $\eta$ are smooth functions.
Indeed,

\[ Q(\hat{g}, \hat{g}) + Q(\hat{g}, g) = \]
\[ \frac{1}{2} \int \eta (\hat{Q} + \hat{Q}) + \frac{1}{2} \int \omega (\hat{Q} + Q) = \]
\[ \frac{1}{2} \int \eta \left( \frac{2\hat{Q} + \hat{P} \eta}{2(Q + P \omega) + P \eta} \right) + \frac{1}{2} \int \omega (2Q + P \omega) = \]
\[ \frac{1}{2} \int (\omega + \eta) (2Q + P(\omega + \eta)) = \]
\[ Q(\hat{g}, g). \]

The underbrace step used conformal invariance, \( \hat{P} = P \). The last step used the formal self-adjointness of \( P \) to equate

\[ 2 \int \eta P \omega \text{ and } \int \eta P \omega + \int \omega P \eta. \]

\( Q(g_1, g_2) \) is a cocycle whose variation (in \( g_1 \), in the \( \omega \) direction) is \( \int \omega Q_1 \).
**Background:** It’s known that there are Polyakov formulas expressing functional determinant quotients within a conformal class as differential polynomials in the conformal factor.

For example, let $Y$ be the conformal Laplacian; then

$$- \log \frac{\det \hat{Y}}{\det Y} = \int_M \omega \text{polyn}(\nabla \cdots \nabla \omega, \nabla \cdots \nabla R) \geq 1$$

$\ominus$ (global term)

in even dims., for $\hat{g} = e^{2\omega}g$. The global term vanishes if $\mathcal{N}(Y) = 0$ (a conformally invt. property); otherwise it records the variation of the global inner product on the null space.
Similarly with $Y$ replaced by anything with decent elliptic and conformal behavior. This includes detour torsion quantities developed in recent joint work with Rod Gover, generalizing Cheeger’s half-torsion.

**Question:** Can the RHS above be expressed more invariantly?

**Answer:** In low even dims. (2,4,6), and conjecturally in all even dims., for $A$ a power of a conformally covariant operator with suitable positive ellipticity properties,

$$- \log \frac{\det \hat{A}}{\det A} = c \int_M \omega(\hat{Q} + Q) + \int_M (\hat{F} - F) + \text{(global term)}$$

for some (universal) constant $c$, where $F$ is some density-valued local invt. (which vary depending on what $A$ is).