Complexity results of path following algorithms for linear programming which take into account the geometry of the central path

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**Goals of the talk**

- Understand the behavior of the central path and the Mizuno-Todd-Ye predictor-corrector (MTY P-C) algorithm for linear programming from the geometric point of view.

- Estimate the iteration complexity of the MTY P-C algorithm in terms of the integral of a certain curvature of the central path.

- Relate the above integral to a new iteration complexity bound for the MTY P-C algorithm involving a certain condition number of the constraint matrix \( A \).
Talk Outline

- LP problem and assumptions;
- central path and its neighborhood;
- Mizuno-Todd-Ye predictor-corrector (MTY P-C) algorithm;
- condition number and scale-invariance;
- iteration complexity bounds for the MTY P-C alg.
  - classical one (1990)
  - new one (2003)
- illustrative LP instance
- curvature of the central path
- iteration complexity bounds in terms of a curvature integral
- directions for future research
**The LP Problem**

(P) \( \text{minimize}_x c^T x \)

subject to \( Ax = b, \ x \geq 0, \)

(D) \( \text{maximize}_{(y,s)} b^T y \)

subject to \( A^T y + s = c, \ s \geq 0, \)

**Assumptions**

1) (P) and (D) have interior-feasible solutions.

2) the rows of the \( m \times n \) matrix \( A \) are linearly independent.

**Definition:** The **duality gap** of a feasible \( w = (x, y, s) \) is given by

\[
\begin{align*}
   c^T x - b^T y &= (A^T y + s)^T x - b^T y \\
   &= x^T s.
\end{align*}
\]
CENTRAL PATH AND ITS NEIGHBORHOOD

For each $\nu > 0$, the system

$$
XSe = \nu e, \\
Ax - b = 0, \quad (x, s) \geq 0, \\
A^T y + s - c = 0,
$$

where $X = \text{Diag}(x)$, $S = \text{Diag}(s)$ and $e = (1, \ldots, 1)^T$, has a unique solution $w(\nu) = (x(\nu), y(\nu), s(\nu))$, which converges to a primal-dual optimal solution as $\nu \to 0$.

The MTY P-C is based on the 2-norm neighborhood of the central path:

$$
\mathcal{N}(\beta) \equiv \{w = (x, y, s) \text{ feasible} : \|Xs - \mu e\| \leq \beta \mu\},
$$

where $\mu = \mu(w) \equiv (x^T s)/n$ and $\beta \in (0, 1)$ is a fixed constant.
$w(\nu) = (x(\nu), s(\nu), y(\nu))$

$\mu(w) := \frac{c^T x - b^T y}{n} = \frac{s^T x}{n}$
$\mu(\mathbf{w}) = 0$

P-D central path

$\mathcal{N}(\beta)$

$\bullet \mathbf{w}^0$

$\bullet \mathbf{w}^1$

$\bullet \mathbf{w}^2$
\[ \mu(w) = 0 \]

\[ \mathcal{N}(\beta^2) \]

\[ \mathcal{N}(\beta) \]

\[ w + \alpha_p \Delta w^a \]

\[ w + \Delta w^c \]

\[ w^+ \]
**Search Directions**

For a strictly feasible $w = (x, y, s)$, the Newton direction $\Delta w = (\Delta x, \Delta y, \Delta s)$ towards the point $w(\nu) = (x(\nu), y(\nu), s(\nu))$ is the solution of

\[
X\Delta s + S\Delta x = -Xs + \nu e \\
A\Delta x = 0 \\
A^T\Delta y + \Delta s = 0
\]

Setting $\nu = 0$ yields the predictor (or affine scaling) direction at $w$.

Setting $\nu = \mu(w)$ yields the corrector (or centrality) direction at $w$. 


An iteration of the MTY P-C Alg.

Let $w = (x, y, s) \in \mathcal{N}(\beta^2)$ be given, where $\beta \in (0, 1/2]$.

1) Compute the AS direction $\Delta w^a = (\Delta x^a, \Delta y^a, \Delta s^a)$ at $w$;

2) Let $\alpha_p > 0$ be the largest $\alpha \in [0, 1]$ such that $w + \alpha \Delta w^a \in \mathcal{N}(\beta)$;

3) Set $w_p = w + \alpha_p \Delta w^a$;

4) Compute the corrector direction $\Delta w^c = (\Delta x^c, \Delta y^c, \Delta s^c)$ at $w_p$;

5) The next point $w^+$ is determined as $w^+ = w_p + \Delta w^c$;

It can be proved that $w^+ \in \mathcal{N}(\beta^2)$. Hence, a new iteration can be started by setting $w \leftarrow w^+$ and going back to 1).
The condition number $\tilde{\chi}_A$

Define

$$\tilde{\chi}_A \equiv \sup\{\|{(ADA^T)^{-1}AD}\| : D \in \mathcal{D}\},$$

where $\mathcal{D}$ denotes the set of all positive definite diagonal matrices.

Facts:

1) \(\tilde{\chi}_A = \max\{\|B^{-1}A\| : B \text{ is a basis of } A\}\).

2) Finding an upper bound for $\tilde{\chi}_A$ is a $\mathcal{NP}$ hard problem.

3) If $A$ integral then $\tilde{\chi}_A \leq 2^{L_A}$, where $L_A$ is the input size of $A$. 
Let $D$ be a positive diagonal matrix and consider the pair of LPs:

\[
\begin{align*}
(P) \quad & \text{minimize} \quad (Dc)^T \tilde{x} \\
& \text{subject to} \quad AD\tilde{x} = b, \quad \tilde{x} \geq 0,
\end{align*}
\]

\[
\begin{align*}
(D) \quad & \text{maximize} \quad b^T \tilde{y} \\
& \text{subject to} \quad DA^T \tilde{y} + \tilde{s} = \tilde{c}, \quad \tilde{s} \geq 0,
\end{align*}
\]

obtained from $(P)$ and $(D)$ by performing the change of variables $(x, y, s) = \Phi(\tilde{x}, \tilde{y}, \tilde{s}) \equiv (D\tilde{x}, \tilde{y}, D^{-1}\tilde{s})$.

The MTY P-C algorithm is scaling-invariant, i.e., if \{w^k\} and \{\tilde{w}^k\} denote the sequence of iterates generated by the MTY P-C algorithm in the original and the scaled space, then $w^k = \Phi(\tilde{w}^k)$ for all $k \geq 1$, as long as $w^0 = \Phi(\tilde{w}^0)$. 
Iteration-Complexity Bounds

Given \(0 < \nu_f < \nu_i\), denote by \(N(\nu_i, \nu_f, \beta)\) the largest possible number of iterations required by the MTY P-C algorithm to find an iterate with duality gap \(\leq \nu_f\) when started from any \(w^0 \in \mathcal{N}(\beta^2)\) such that \(\mu(w^0) = \nu_i\).

**Classical Result:** For any \(\beta \in (0, 1/2]\),

\[
\sqrt{\beta} \cdot N(\nu_i, \nu_f, \beta) \leq \sqrt{n} \log \left( \frac{\nu_i}{\nu_f} \right)
\]

**Lemma:** Suppose \(w \in \mathcal{N}(\beta^2)\), where \(\beta \in (0, 1/2]\). Then, \(w^+ \in \mathcal{N}(\beta^2)\) and

\[
\frac{\mu(w^+)}{\mu(w)} \leq 1 - \sqrt{\frac{\beta}{n}}
\]
Vavasis-Ye Algorithm

**Iteration Complexity Bound:** The number of iterations to solve a linear program is

\[ O(n^{3.5} \log(n + \bar{\chi}_A)) \]

**Note:** Their bound does not depend on \( \nu_i \) and \( \nu_f \)!

Their algorithm accelerates an ordinary primal-dual path following method (e.g., the MTY P-C algorithm) by using from time to time a step called the *layered-least-square step.*

V-Y algorithm is not scaling invariant.
New complexity for the MTY method

Theorem (Monteiro and Tsuchiya 2003): For any $\beta \in (0, 1/2]$,  

$$N(\nu_i, \nu_f, \beta) = \mathcal{O}(T(\nu_i/\nu_f) + n^{3.5} \log(\bar{\chi}_A^* + n))$$

iterations, where $\bar{\chi}_A^* \equiv \inf\{\bar{\chi}_{AD} : D \in \mathcal{D}\}$ and  

$$T(\eta) \equiv \min \{ n^2 \log (\log \eta), \log \eta \}$$

Remark: In contrast to $\bar{\chi}_A$, the quantity $\bar{\chi}_A^*$ is scaling invariant. Usually $\bar{\chi}_A^* \ll \bar{\chi}_A$. Hence, the above complexity is not comparable to the one associated with the V-Y method.

Lemma: For any $\beta \in (0, 1/2]$ and $w \in \mathcal{N}(\beta^2)$:  

$$\frac{\mu(w^+)}{\mu(w)} \leq \frac{\kappa(w)^2}{\beta},$$

where  

$$\kappa(w) := \left( \frac{\|\Delta x^a(w)\Delta s^a(w)\|}{\mu(w)} \right)^{1/2}$$
Consequences

Under the Turing machine model, the iteration-complexity of the MTY P-C algorithm is

\[ \mathcal{O}(n^{3.5}L_A + \min\{L, n^2 \log L\}) \]

\[ \leq \mathcal{O}(n^{3.5}L_A + L) \]

Given \( A \), there exist many nontrivial \((b, c)\) for which the complexity of the MTY P-C algorithm for solving \((P)\) and \((D)\) is \( \mathcal{O}(L) \)
**Example**

Consider the LP

$$\max \{ b^T y : A^T y \leq c \},$$

where

$$A = \begin{pmatrix}
0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & 0 \\
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\
-1 & \frac{1}{3} & 1 & \frac{1}{3}
\end{pmatrix},$$

$$b = \begin{pmatrix}
-10^{-9} \\
-10^{-5} \\
-1
\end{pmatrix}, \quad c = \begin{pmatrix}
0 \\
\frac{2\sqrt{6}}{3} \\
0 \\
0
\end{pmatrix}.$$
Example (continued)

Figure 1: Figure for the LP instance
Example (continued)

![Graph showing log \( \mu \) versus \( N(\nu_i, \mu, \beta) \).](image)

**Figure 2**: \( \log \mu \) versus \( N(\nu_i, \mu, \beta) \) (\( \cdot : \sqrt{\beta} = 0.0025; + : \sqrt{\beta} = 0.005; * : \sqrt{\beta} = 0.01; \circ : \sqrt{\beta} = 0.02 \))
Figure 3: $\log \mu$ versus $\sqrt{\beta} \cdot N(\nu_i, \mu, \beta)$ ($\cdot : \sqrt{\beta} = 0.0025; + : \sqrt{\beta} = 0.005; * : \sqrt{\beta} = 0.01; \circ : \sqrt{\beta} = 0.02$)

**Question:** Does $\sqrt{\beta} \cdot N(\nu_i, \mu, \beta)$ always converge as $\beta \to 0$?
Figure 4: $\log \mu$ versus $\sqrt{\beta} \cdot N(\nu_1, \mu, \beta)$ (The big dots correspond to the ones in Figure 1.)

**Question:** How to define straight and curved parts of the central path?
**Curvature of the central path**

**Definition:** The curvature of the central path is the function \( \kappa : (0, \infty) \to [0, \infty) \) defined as

\[
\kappa(\nu) \equiv \|\nu \dot{x}(\nu) \dot{s}(\nu)\|^{1/2}, \quad \forall \nu > 0.
\]

**Note:** if \( w = w(\nu) \) then \( \kappa(w) = \kappa(\nu) \)

For a given \( \nu > 0 \) and \( \beta \in (0, 1) \), define

\[
\mathcal{T}(\beta, \nu) \equiv \{ t \in \mathbb{R} : w(\nu) - t \nu \dot{w}(\nu) \in \mathcal{N}(\beta) \}
\]

Note that \( w(\nu) - t \nu \dot{w}(\nu) \approx w((1 - t) \nu) \).

**Proposition:** \( \mathcal{T}(\beta, \nu) \) is a closed interval and

\[
\lim_{\beta \downarrow 0} \frac{\text{length of } \mathcal{T}(\beta, \nu)}{\sqrt{\beta}} = \frac{2}{\kappa(\nu)}
\]
**Complexity in terms of the curvature**

Theorem (Sonnevend, Stoer and Zhao 1994):

\[
N(\nu_i, \nu_f, \beta) = O \left( \int_{\nu_f}^{\nu_i} \frac{\kappa(\nu)}{\nu} d\nu + \log \left( \frac{\nu_i}{\nu_f} \right) \right).
\]

**Note:** Since \( \kappa(\nu) \leq \sqrt{\frac{n}{2}} \) for all \( \nu > 0 \), the classical bound follows from the above bound.

**Theorem 1 (Monteiro and Tsuchiya 2005):**

\[
\lim_{\beta \to 0} \sqrt{\beta} \cdot N(\nu_i, \nu_f, \beta) = \int_{\nu_f}^{\nu_i} \frac{\kappa(\nu)}{\nu} d\nu \leq \sqrt{n} \log \left( \frac{\nu_i}{\nu_f} \right)
\]

Recall that one of the M-T bounds is

\[
N(\nu_i, \nu_f, \beta) = O \left( n^{3.5} \log(\bar{\chi}_A^* + n) + \log \left( \frac{\nu_i}{\nu_f} \right) \right).
\]
Bound on the curvature integral

Theorem 2 (Monteiro and Tsuchiya 2005):
For every $0 < \nu_f < \nu_i$, we have:

$$\int_{\nu_f}^{\nu_i} \frac{\kappa(\nu)}{\nu} \, d\nu \leq O\left( n^{3.5} \log(\bar{\chi}_A^* + n) \right)$$

Hence,

$$\int_{0}^{\infty} \frac{\kappa(\nu)}{\nu} \, d\nu \leq O\left( n^{3.5} \log(\bar{\chi}_A^* + n) \right)$$
Vavasis and Ye 1996: ”The central path consists of $O(n^2)$ long and straight parts and other curved parts”

We want to formally establish this statement!

**Theorem 3:** For any $\kappa \in (0, \sqrt{n/2})$, there exist $l \leq n(n - 1)/2$ closed intervals $I_k$ such that:

a) $\{\nu > 0 : \kappa(\nu) \geq \kappa\} \subseteq \bigcup_{k=1}^{l} I_k$

(union of $I_k$’s covers portion with large curvature)

b) the logarithmic length of each $I_k$ is bounded by $O\left(n \log\left(\chi^*_A + n\right) + n \log \kappa^{-1}\right)$

(independent of b and c)
The blue parts are long but quite straight!

The MTY P-C algorithm converges $R$-quadratically over the blue parts.

There are at most $O(n^2)$ blue and green parts.
Directions for future research

- Generalizations to other cone programming problems such as SOCP and SDP
- Are infeasible path following methods ammenable to the same kind of analysis? Can new iteration complexity bounds be obtained for them?
- Is it possible to interpret the curvature $\kappa(\nu)$ as the one used in differential geometry? What further insights can be gained through this approach?
- Can an iteration complexity bound depending only on $n$ and $\bar{\chi}_A^*$ be derived for the MTY P-C algorithm?
- Is it possible to derive a Zhao and Stoer’s type result with $\log \log$, i.e.

$$N(\nu_i, \nu_f, \beta) = O \left( \int_{\nu_f}^{\nu_i} \frac{\kappa(\nu)}{\nu} d\nu + n^2 \log \log \left( \frac{\nu_i}{\nu_f} \right) \right).$$